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A NOTE ON HAMILTON’S EQUATIONS AND INVARIANT IMBEDDING

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This Memorandum results from RAND's continuing study of computational aspects of the solution of the equations of mathematical physics. A new basic partial differential equation of particle mechanics is derived.
SUMMARY

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q,p)$. The equations of motion are

\begin{align*}
(1) & \quad \dot{q} = H_p, \\
(2) & \quad -\dot{p} = H_q, \quad 0 \leq t \leq T.
\end{align*}

As boundary conditions we specify

\begin{align*}
(3) & \quad q(0) = 0, \\
(4) & \quad p(T) = c.
\end{align*}

If our aim is to solve the system of equations (1) and (2), subject to the boundary conditions in (3) and (4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time $T$, $q(T)$, so that we shall have a complete set of conditions at time $T$. This would enable us to integrate the system (1) and (2) numerically, subject to initial values. Toward this end we introduce the function

\begin{equation}
(5) \quad r(c,T) = \text{the displacement at time } T, \text{ the initial displacement being zero and the momentum at time } T \text{ being } c.
\end{equation}

We derive a partial differential equation for this function.
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A NOTE ON HAMILTON'S EQUATIONS AND INVARIANT IMBEDDING

1. INTRODUCTION

Consider the motion of a particle on a line where, as usual, we characterize the process by a Hamiltonian $H = H(q,p)$. The equations of motion are

\begin{align}
(1.1) \quad \dot{q} &= H_p, \\
(1.2) \quad -\dot{p} &= H_q, \quad 0 \leq t \leq T.
\end{align}

As boundary conditions we specify

\begin{align}
(1.3) \quad q(0) &= 0, \\
(1.4) \quad p(T) &= c.
\end{align}

If our aim is to solve the system of equations (1.1) and (1.2), subject to the boundary conditions in (1.3) and (1.4), using a digital computer, we face the well-known difficulties of solving nonlinear two-point boundary-value problems.

It is tempting to try to determine the unknown displacement at time $T$, $q(T)$, so that we shall have a complete set of conditions at time $T$. This would enable us to integrate the system (1.1) and (1.2) numerically, subject to initial values. Toward this end we introduce the function
(1.5) \( r(c,T) = \) the displacement at time \( T \), the initial displacement being zero and the momentum at time \( T \) being \( c \).

We wish to derive an equation for \( r(c,T) \).

2. INVARIANT IMBEDDING

In earlier papers \([1,2]\), we showed the following:

If

\[
\begin{align*}
(2.1) \quad & \frac{du}{dz} = F(u,v), \quad u(0) = 0, \\
(2.2) \quad & -\frac{dv}{dz} = G(u,v), \quad v(x) = c,
\end{align*}
\]

for

\[
(2.3) \quad 0 \leq z \leq x,
\]

and if

\[
(2.4) \quad r(c,x) = u(z)\bigg|_{z=x},
\]

then the function \( r(c,x) \) satisfies the partial differential equation

\[
(2.5) \quad r_x = F(r,c) + G(r,c)r_c
\]

and the initial condition

\[
(2.6) \quad r(c,0) = 0.
\]
Upon applying this result to the system of equations in (1.1) and (1.2), we find that the function \( r(c,T) \) satisfies the equation

\[
\frac{\partial r}{\partial T} = H_p(r,c) + H_q(r,c) \frac{\partial r}{\partial c},
\]

or, more elegantly,

\[
\frac{\partial r}{\partial T} = \frac{\partial H(r,c)}{\partial c}.
\]

This is the desired result, apparently a new equation of mechanics.

3. AN EXAMPLE

Consider the case of harmonic oscillations for which

\[
H(q,p) = \frac{p^2}{2m} + \frac{1}{2} kq^2,
\]

and

\[
q(0) = 0, \quad r(T) = c.
\]

For the unknown displacement at time \( T \), \( r(c,T) \), equation (2.8) becomes

\[
\frac{\partial r}{\partial T} = \frac{\partial}{\partial c} \left[ \frac{c^2}{2m} + \frac{1}{2} kr^2 \right] = \frac{c}{m} + kr \frac{\partial r}{\partial c}.
\]

The solution of this equation, subject to the condition (2.6) is
(3.4) \[ r(c, T) = \frac{c}{\sqrt{\text{cm}}} \tan \left( \frac{\sqrt{E}}{m} T \right). \]

This provides the desired displacement at time \( T \) and also shows that the function \( r(c, T) \) may become infinite for a finite value of \( T \).

4. DISCUSSION

The previous discussion may be generalized to the case of many particles moving in three-dimensional space using the theorem in [1]. Furthermore, various combinations of the q's and p's may be specified at the ends. There are immediate applications to perturbation analysis of results of this nature; see [3].
REFERENCES

