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ON CONTROL PROBLEMS WITH BOUNDED STATE VARIABLES

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PREFACE

The Memorandum is part of the continuing RAND basic research in the mathematics of control problems. These problems are a common mathematical formulation of situations that arise in diverse areas. Common examples occur in the control of aircraft and missile regimes, control of chemical reactors, and control of inventories. The present Memorandum treats a class of control problems in which constraints involving the state variable but not explicitly involving the control variable are present.
SUMMARY

This Memorandum studies control problems in which constraints that are independent of the control are imposed on the state variables. Necessary conditions that hold along an optimal trajectory are derived in a relatively simple fashion from known results in the calculus of variations. These results had been obtained by Gambrelidze in a different fashion.
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ON CONTROL PROBLEMS WITH BOUNDED STATE VARIABLES

1. INTRODUCTION

In a recent paper [1] we showed, among other things, how a fairly general control problem, or programming problem, with constraints can be reduced to a special type of Bolza problem in the calculus of variations. Necessary conditions for the Bolza problem were then translated into necessary conditions for optimal control. These conditions include the maximum principle of Pontryagin [3], [6] for this class of problems and some of the later results of Gamkrelidze [4]. Our results in [1] do not, however, apply to control problems with constraints on the state variables that do not explicitly involve the control variable. Such problems were treated by Gamkrelidze in [4], who modified the arguments in [3] to account for the additional constraints. In this memorandum we shall use the methods of [1] to study such problems, and we shall obtain the results of Gamkrelidze in a very simple fashion from relevant results in the calculus of variations.

2. FORMULATION OF PROBLEM

We assume that the reader is familiar with [1], and we shall use the notation of [1]. Let \( \theta \) be a function of class \( C'' \) on the region \( \mathcal{O} \) of \( (t,x) \)-space such that
the relation \( \theta(t,x) = 0 \) defines a manifold \( \mathcal{B}^* \) which divides \( \mathcal{D} \) into two regions. Let \( \mathcal{B} \) be that subset of \( \mathcal{D} \) defined by the relation

\[
(2.1) \quad \theta(t,x) \geq 0.
\]

We shall consider the control problem as formulated in [1] with \( g = 0 \), and with the added restriction that the curves \( K \) resulting from controls \( u \) in \( \mathcal{A} \) must lie in \( \mathcal{B} \). That is, we consider the following problem.

**Problem I.** Find an element \( u^* \) in the class of admissible controls \( \mathcal{A} \) that minimizes the functional

\[
J(u) = \int_{t_0}^{t_1} f(t,x,u) \, dt.
\]

Here, the state of the system is determined by the system of differential equations

\[
x' = G(t,x,u), \quad x(t_0) = x_0;
\]

the controls and the state of the system satisfy the constraints

\[
R(t,x,u) \geq 0, \quad \theta(t,x) \geq 0;
\]

and the right-hand end point \( (t_1, x_1) \) of the trajectory is a point of a \( p \)-dimensional manifold, \( 0 \leq p \leq n \).

It will be clear from what follows and from [1] how one handles the case in which \( \mathcal{B} \) is a closed region in
with piecewise smooth boundary, each piece of which is defined by a relation \( \theta_i(t,x) = 0 \).

The assumption that \( g = 0 \) is made in order to simplify a certain portion of the argument below. No loss of generality will result, for the control problem of [1] is equivalent to a control problem in which \( g = 0 \) and the initial point lies on a line, as the following transformation shows. Let a new coordinate \( x^{n+1} \) be introduced by means of the following differential equation and end conditions:

\[
\frac{dx^{n+1}}{dt} = 0, \quad x_1^{n+1} = \frac{g(\sigma)}{t_1(\sigma) - t_0}, \quad x_0^{n+1} \text{ free.}
\]

Let the functional to be minimized be

\[
J(u) = \int_{t_0}^{t_1} (f + x^{n+1})dt.
\]

In Problem I we assumed, as we did in [1], that the left-hand end point \((t_0,x_0)\) is fixed. If \((t_0,x_0)\) is constrained to lie on a \(p\)-dimensional manifold \((1 \leq p \leq n)\) in \((t,x)\)-space, the analysis that follows requires the introduction of a transversality condition for the left-hand end point. We leave this to the reader.

We cannot treat Problem I by simply adjoining the constraint (2.1) as an \((r + 1)\)-st component to the constraint vector \( R(t,x,u) \) and then proceeding to use
the analysis of [1]. The reason is that since θ is 
independent of u, the constraint vector (R, θ) will not 
satisfy the constraint condition (2.2)-(ii) of [1] at any 
point of  \( \mathcal{B}^* \), the manifold defined by \( \theta(t,x) = 0 \). The 
constraint conditions for Problem I are the following, 
which are modifications of those in [1]:

(i) If \( r > m \), then at each point of \( \mathcal{J} = \mathcal{B} \times \mathcal{U} \), 
at most \( m \) components of \( R \) can vanish.

(2.2) If \( r \geq m \), then at each point of \( \mathcal{J}^* = \mathcal{B}^* \times \mathcal{U} \), 
at most \((m-1) \) components of \( R \) can vanish.

(ii) At each point of \( \mathcal{J} \) the matrix \( \left( \partial R_i / \partial u^j \right) \), 
where \( i \) ranges over those indices such that 
\( R_i(t,x,u) = 0 \), and \( j = 1, \ldots, m \), has maximum 
rank. At each point of \( \mathcal{J}^* \), if the \( m \)-dimensional 
row vector \( \theta_x \mathcal{C}_u \) is adjoined to this matrix, 
the resulting matrix has maximum rank.

3. EQUIVALENT LAGRANGE PROBLEM

Let \( \eta \) be a scalar, and let

\[
\gamma(t,x,\eta) = \begin{cases} 
\eta^4 - \theta(t,x) & \text{if } \eta \geq 0, \\
\theta(t,x) & \text{if } \eta \leq 0.
\end{cases}
\]

(3.1)

The function \( \gamma(t,x,\eta) \) is clearly \( C^2 \) on the region of 
\( (t,x,\eta) \)-space which is the Cartesian product of the 
\( \eta \)-axis and \( \mathcal{J} \). If we let

\[ y' = u, \quad y(t_0) = 0, \]


then Problem I is readily seen to be equivalent to the following problem of Lagrange in \((n + m + r + 2)\)-dimensional \((t,x,y,\xi,\eta)\)-space.

**Problem II.** Find an arc \((x(t),y(t),\xi(t),\eta(t))\) that minimizes

\[
\int_{t_0}^{t_1} f(t,x,y')\,dt
\]

in the class of arcs that are piecewise \(C^n\) and that satisfy the differential equations

\[
G(t,x,y') - x' = 0,
\]
\[
R(t,x,y') - (\xi')^2 = 0,
\]
\[
\xi_t + \xi_x x' - \gamma \eta \eta' = 0,
\]

and also the end conditions

\[
x(t_0) = x_0, \quad y(t_0) = 0, \quad \xi(t_0) = 0, \quad \gamma(t_0,x_0,\eta_0) = 0,
\]
\[
t_1 = t_1(\sigma), \quad x_1 = x_1(\sigma), \quad \gamma(t_1,x_1,\eta_1) = 0,
\]
\[
\xi_1 = \xi(t_1), \text{ free, } \quad y_1 = y(t_1), \text{ free.}
\]

Note that the last equation in (3.3) and the end conditions imply that \(\gamma(t,x,\eta) = 0\) along each arc. Hence, by (3.1), \(\theta \geq 0\) along each arc.

Let us suppose that \(u^*\) is an optimal control in \(A\), and let \(K^*\) be the corresponding curve in \((t,x)\)-space. Let \(K^*_1\) be the corresponding arc in \((t,x,y,\xi,\eta)\)-space.
Consider the following \((n + r + 1)\) by \((m + n + r + 1)\) matrix along \(K^*_1\):

\[
\begin{pmatrix}
G_y' & -I & 0 & 0 \\
R_y' & 0 & -2\Xi' & 0 \\
0 & \theta_x & 0 & -\gamma_\eta
\end{pmatrix}
\tag{3.6}
\]

where \(2\Xi'\) is an \(r+r\) diagonal matrix with entries \(2(\xi^i)\)' on the diagonal. The rank of (3.6) is clearly the same as the rank of the matrix

\[
\begin{pmatrix}
G_y' & -I & 0 & 0 \\
R_y' & 0 & -2\Xi' & 0 \\
\theta_x G_y' & 0 & 0 & -\gamma_\eta
\end{pmatrix}
\tag{3.7}
\]

Since \(\gamma = 0\) along \(K^*_1\), it follows from (3.1) that \(\gamma_\eta > 0\) if \(\theta > 0\), and that \(\gamma_\eta = 0\) if \(\eta \leq 0\). Hence, using the constraint conditions (2.2) and arguments similar to those used to determine the rank of (3.8) in [1], we see that the matrix (3.7) has rank \((n + r + 1)\) at \(r^*_1\) points of \(K^*_1\). Hence (3.6) has rank \((n + r + 1)\).

The above argument is not restricted to \(K^*_1\); it shows that (3.6) has rank \((n + r + 1)\) at all elements \((t,x,y,\xi,\eta,x',y',\xi',\eta')\) of a curve for which (3.3) and the end conditions (3.4) and (3.5) hold.

If \(t_1(\sigma)\) and \(x_1(\sigma)\) define a \(p\)-dimensional terminal manifold for Problem I, then the right-hand end conditions for Problem II and the restriction that
(t_1, x_1) is in \( D \) determine a \( p \)-dimensional terminal manifold in \( (t, x, \eta) \)-space for Problem II. We suppose that in a neighborhood of the right-hand end points of \( K_1^* \), this manifold is given parametrically as follows:

\[
t_1 = t_1(s), \quad x_1 = x_1(s), \quad \eta_1 = \eta_1(s).
\]

We note that

\[
(3.8) \quad \gamma(t_1(s), x_1(s), \eta_1(s)) = 0.
\]

Since \( u^* \) is an optimal control, it is clear that \( K_1^* \) furnishes a minimum for Problem II. From this and from the preceding discussion it follows that the multiplier rule, the Weierstrass condition, and the Clebsch condition as given in Bliss [2] and extended by McShane [5] hold along \( K_1^* \). The function \( F \) in this instance is defined as follows:

\[
(3.9) \quad F = \psi_0 f + \psi(G - x') + \mu(R - (\xi')^2) + \nu(\theta e + \theta x' - \gamma \eta').
\]

From the Euler equations

\[
\frac{dF_{y'}}{dt} = F_y, \quad \frac{dF_{\xi'}}{dt} = F_\xi,
\]

the continuity of \( F_{y'} \) and \( F_{\xi'} \), and the relations

\[
(3.10) \quad F_{y'} = 0, \quad F_{\xi'} = 0 \text{ at } (t_1, x_1, \eta_1),
\]

which we obtain from the transversality condition, we deduce as we did in [1] that along \( K_1^* \).
From the Euler equation \( \frac{dF}{dt} = F \) we find that between corners of \( K_1^* \),

\[
(3.12) \quad v' \gamma = 0 .
\]

The Euler equation \( \frac{dF}{dt} = F \) takes the form

\[
(3.13) \quad \frac{d}{dt} (-v + \gamma' \theta) = \psi_0 f_x + \psi G_x + \mu R_x + \nu(\theta_{tx} + \theta_{xx}x').
\]

If we make use of (3.10) in the remaining equations of the transversality condition, we get

\[
(F - x' F_x' - \gamma' F_\eta')t_{ls} + F_x'x_{ls} + F_\eta'\gamma_{ls} = 0 .
\]

From (3.3) we obtain that \( x' = G \) and that \( F = \psi_0 f \) at \( (t_1', x_1', \eta_1') \). Using this and the last equation in (3.3), we can rewrite the preceding equation as follows:

\[
(\psi_0 f + \psi G)t_{ls} - \psi x_{ls} + \nu(\theta_t t_{ls} + \theta_x x_{ls} - \gamma_{ls}) = 0 .
\]

Hence, using (3.8) we find that the transversality condition, in addition to yielding (3.10), gives

\[
(3.14) \quad (\psi_0 f + \psi G)t_{ls} - \psi x_{ls} = 0 .
\]
Another necessary condition is the continuity along \( K_1^* \) of the expression

\[
F - x' F_x' - y' F_y' - \xi F_{x'} - \eta F_{y'}.
\]

If we take (3.3) and (3.10) into account, this expression may be rewritten as follows along \( K_1^* \):

\[
(3.15) \quad \psi_0 f + \psi G + \nu \theta_t.
\]

Using arguments similar to those used above and in [1], we may rewrite the Weierstrass condition in the following form:

\[
(3.16) \quad \psi_0 (f(t,x,y') - f(t,x',y')) + \psi (x' - x) \geq 0.
\]

From the Clebsch condition we deduce, as we did in [1], that along \( K_1^* \),

\[
(3.17) \quad \mu \leq 0,
\]

and that

\[
(3.18) \quad e((\psi_0 f + \psi G + \mu R)_{y'y'}) \geq 0
\]

for all \( m \)-dimensional solution vectors \( e \) of the following linear systems: \( \hat{R}_{y'y} e = 0 \) at points that are interior to \( \mathcal{S} \); \( \hat{R}_{y'y} e = 0 \) and \( \theta G_{x'y} e = 0 \) at points of \( K_1^* \) that correspond to points of \( \mathcal{S}^* \). The vector \( \hat{R} \) is obtained from \( R \) by taking those components of \( R \) that vanish at the point.
4. AN INTERIOR SEGMENT

We now consider the curve $K^*$ corresponding to the optimal control $u^*$. We adopt Gamkrelidze's definition [4] and say that the point $(\tau, x(\tau))$ of $K^*$ is a junction point if $t_0 < \tau < t_1$ and if there exists a $\delta > 0$ such that either the segment of $K^*$ for which $\tau - \delta < t < \tau$, or the segment for which $\tau < t < \tau + \delta$ (or both), lies in the interior of $\mathcal{S}$. We call $\tau$ a junction time. We suppose that $K^*$ has a finite number of junction points. Explicitly, we suppose that if $\tau$ is the largest junction time, then the portion of $K^*$ defined for $\tau < t < t_1$ is interior to $\mathcal{S}$. We denote this segment by $K_A^*$ and also use this notation for the corresponding segment of $K_1^*$.

Since $K_A^*$ is interior to $\mathcal{S}$, it follows that $\eta > 0$. Hence from (3.1) we have $\gamma \eta > 0$, and so from (3.12) we get that between corners of $K_A^*$, $\nu$ is constant. Moreover, the end condition (3.14) places no restriction on $\nu(t_1)$. Since $\nu$ is constant between corners of $K_1^*$, equation (3.13) can be written as

$$\phi' = - (\psi_0 f_x + \psi G_x + \mu R_x).$$

Note that the differential equation (4.1) and the end conditions (3.14) are independent of the values assigned to $\nu$. Using this observation and the fact that the left-hand end point of $K_1^*$ is fixed, we can see
by examining the proof in [5] that we can always choose $(\psi_0, \psi, \mu) \neq 0$ at the right-hand end point $(t_1, x_1)$ of $K_1^*$. From the constraint condition (2.2)-(ii) and from (3.11) it follows that $\mu$ is determined uniquely as a linear function of $(\psi_0, \psi)$ on the interval $\tau \leq t \leq t_1$. Hence, on this interval, (4.1) can be written as a linear differential equation in $\psi$, and so if $(\psi_0, \psi) = 0$ at any point of $K_A^*$, then $(\psi_0, \psi) = 0$. Moreover, in this event $\mu = 0$. Since $(\psi_0, \psi, \mu) \neq 0$ at $(t_1, x_1)$, we can therefore conclude that $(\psi_0, \psi) \neq 0$ at every point of $K_A^*$.

Define

$$(4.2) \quad H(t, x, u, \lambda_0, \lambda) = \lambda_0 f + \lambda G.$$ 

For $\tau < t < t_1$, let

$$(4.3) \quad \lambda_0 = \psi_0, \quad \lambda = \psi,$$

and take $v$ to be a constant on the entire interval $\tau < t < t_1$. ($K_1^*$ may have corners in this interval.) It follows from (4.1)-(4.3), (3.11), (3.14)-(3.18), and from the relation $y' = u$, that Theorem 2 of [1] holds along $K_A^*$.

5. A BOUNDARY SEGMENT

Let $\tau'$ be the largest of the junction times that are less than $\tau$. If there are none, take $\tau' = t_0$. We next suppose that the segment of $K^*$ defined for $\tau' \leq t \leq \tau$ lies entirely in $\mathcal{B}^*$. We denote this
segment (and the corresponding segment of \( K_1^* \)) by \( K_B^* \).

To simplify the exposition we shall suppose that \( K_B^* \) has no corners and the same components of \( R \) vanish all along \( K_B^* \). If the contrary holds, the argument requires trivial modifications, which we leave to the reader.

Let

\[
\phi(t,x,u) = \theta_t + \theta_x G.
\]

Then along \( K_B^* \), we clearly have

\[
\phi = 0.
\]

On the interval \( \tau' \leq t \leq \tau \), let

\[
\lambda_0 = \psi_0, \quad \lambda = \psi - \nu \theta_x.
\]

If we substitute (5.3) into (3.13) and use the definition of \( \phi \) given in (5.1) and the definition of \( H \) given in (4.2), we can rewrite (3.13) as

\[
\frac{d\lambda}{dt} = -(H_x + \mu R_x + \nu \phi_x).
\]

If we substitute (5.3) into the second equation of (3.11) and replace \( y' \) by \( u \), we can rewrite this equation as follows:

\[
H_u + \mu R_u + \nu \theta_x G_u = 0.
\]

Similarly, the substitution of (5.3) into the Weierstrass condition (3.16) leads to the relation
\[ H(t, x, y') - H(t, x, y') + v \theta_x (x' - x') \geq 0 \]

for all admissible \((t, x, x', y')\). Since the element 
\((t, x, x', y')\) is admissible, it satisfies the last equation 
of (3.3), and so \(\theta_x (x' - x') = \theta_t + \theta_t = 0\). Hence, 
setting \(u = y'\), we can rewrite (3.16) as

\[ (5.6) \quad H(t, x, u) \geq H(t, x, u^*) \]

along \(K^*_B\) for all admissible \(u\) such that \(\phi(t, x, u) = 0\).

Finally, equation (3.18) of the Clebsch condition becomes

\[ (5.7) \quad e((H + \mu R + v \delta_x G)_{uu}) e \geq 0 . \]

From the necessary conditions for Problem II, it follows that there exists a constant \(\lambda_0 \geq 0\) and

functions \((\lambda, \mu, v)\) such that (5.4) - (5.7) hold along \(K_B^*\). From the constraint condition (2.2)-(ii), it follows 
that we may solve (5.5) uniquely for \((\mu, v)\) as linear 
functions of \((\lambda_0, \lambda)\). Substitution of this solution into 
(5.4) yields a system of linear differential equations 
for \(\lambda\). Hence if \((\lambda_0, \lambda)\) is determined at a point of 
\(K_B^*\), then the solution \(\lambda\) of (5.4) and the functions

\((\mu, v)\) are uniquely determined along \(K_B^*\).

From the Weierstrass-Erdmann corner condition for Problem II, we see that

\[ (5.8) \quad F_x' = -\psi + v \theta_x \]
is continuous at the junction point \((\tau, x(\tau))\). Thus, if we denote functions along \(K_A^*\) by a subscript \(A\) and functions along \(K_B^*\) by a subscript \(B\), then at \((\tau, x(\tau))\) we have

\[
(5.9) \quad -\psi_A^+ + \nu_A^+ \theta_x = -\psi_B^- + \nu_B^- \theta_x.
\]

Using (4.3) and (5.3), this can be rewritten as

\[
(5.10) \quad \lambda_B^- = \lambda_A^+ - \nu_A^+ \theta_x, \quad \nu_A^+ \text{ arbitrary.}
\]

This relation and the relation

\[
(5.11) \quad \lambda_{0B} = \lambda_{0A} = \psi_0,
\]

which follows from (4.3), (5.3), and the constancy of \(\psi_0\), therefore serve to determine \((\lambda_0, \lambda, \mu, \nu)\) uniquely along \(K_B^*\).

From the continuity of (3.15), and from (4.3), (5.3), and (5.2), we readily find that at the junction point \((\tau, x(\tau))\),

\[
(5.12) \quad \theta_t = \psi^+ + \nu_A \theta_t.
\]

With the help of (5.10) and (5.2), this can be rewritten as follows:

\[
(5.13) \quad 
\begin{align*}
\lambda_0 (f^- - f^+) + \lambda^+ (G^- - G^+) &= 0, \\
\lambda_0 (f^- - f^+) + \lambda^- (G^- - G^+) &= \nu_A^+ (\theta_x G^+ + \theta_t).
\end{align*}
\]
It is readily verified that the vector

\[(5.14) \quad (\lambda_0, \lambda_B, \mu_B, \nu_B) = (0, \rho \theta_x, 0, -\rho),\]

where \(\rho\) is any real constant, satisfies (5.4) and (5.5) along any curve \(K\) obtained from an admissible control and lying in \(\mathcal{G}^*\). Substitution of (5.14) into the relation (5.6) reduces (5.6) to the identity \(\rho \theta_t = \rho \theta_t\) along any such curve. From (5.5) and the constraint condition (2.2)–(ii) it follows that (5.14) is the unique solution of (5.4)–(5.6) with \(\lambda_0 = 0, \lambda = \rho \theta_x, \rho\) arbitrary. It is immediate from (5.3) that for Problem II,

\[(5.15) \quad (\psi_0, \psi_B, \mu_B, \nu_B) = (0, 0, 0, -\rho), \quad \rho\) arbitrary,

is the unique vector corresponding to (5.14). The vector (5.15) reduced the Euler equations (3.13) and (3.11), and the Weierstrass condition (3.16) to identities for all admissible curves lying in \(\mathcal{G}^*\). We shall refer to (5.14) or (5.15) as a trivial multiplier vector. Note that since \(\rho\) is arbitrary, the zero vector is included.

If we have

\[(5.16) \quad \lambda_0^+ = 0, \quad \lambda_A^+ = k \theta_x, \quad k \neq 0,\]

at \((\tau, x(\tau))\), for every set of multipliers \((\lambda_0, \lambda_A, \mu_A, \nu_A)\) such that equations (5.4)–(5.7) hold along \(K_A^*\), then it follows from (5.10) and the discussion of the preceding paragraph that along the segment \(K_B^*\) we obtain the trivial multipliers (5.14). To avoid this, we proceed
as follows. We consider a Problem I', which we define as Problem I with fixed initial point \((\tau', x(\tau'))\) and fixed terminal point \((\tau, x(\tau))\). It is clear that the segment \(K_B^*\) must furnish a relative minimum for Problem I'. For, if some other curve \(K'\) furnished a minimum, then we could replace the segment \(K_B^*\) of \(K^*\) by \(K'\) and thereby contradict the minimality of \(K^*\). Since the Lagrange problem corresponding to Problem I' has fixed end points, it can be seen from the proof in [5] of the necessary conditions that we may take \((\psi_0, \psi(\tau_2)) \neq 0\).

Hence there exist nontrivial multipliers \((\psi_0, \psi, \mu, \nu)\) such that (5.4)-(5.7) hold along \(K_B^*\), even if (5.16) holds. In this event, however, (5.10)-(5.13) are no longer valid.

The following theorem is a consequence of the preceding discussion.

**Theorem 1.** On the interval \([\tau', \tau]\) there exists a constant \(\lambda_0 \geq 0\) and a continuous \(n\)-dimensional vector \(\lambda(t)\) such that \((\lambda_0, \lambda) \neq (0, \rho \theta_x), \rho\) arbitrary; an \(n\)-dimensional vector \(\mu(t) \leq 0\), continuous except perhaps at values of \(t\) corresponding to corners of \(K_B^*\); and a function \(\nu(t)\) with the same continuity properties as \(\mu\), such that along \(K_B^*, (5.4)-(5.7)\) hold.

At the junction point \((\tau, x(\tau))\), either (5.6) holds for every \((\lambda_{0A}, \lambda_A^+)\), or the jump conditions (5.10) and (5.12) (and hence (5.13)) hold.
Remark 1. This result was obtained by Gamkrelidze [4], who used different arguments. He also presents a result that in our notation reads $dv/dt \geq 0$ along $K_A^*$.  

Remark 2. In Sec. 4 we assumed that the segment $K_A^*$ was interior to $\mathcal{I}$. If we had assumed that the segment $K_A^*$ was in $\mathcal{I}^*$, then we would still conclude that, except for the jump conditions, Theorem 1 holds along $K_A^*$. This follows from the arguments used to establish the theorem and the observation that the transversality condition (3.14) places no restriction on $\nu(t_1)$.

6. CONCLUSION

If $\tau' > t_0$, let us suppose that $\tau''$ is the largest junction time that is less than $\tau'$; if there is no such junction time, take $\tau'' = t_0$. The segment of $K$ defined for $\tau'' < t < \tau'$ then lies entirely within $\mathcal{I}$. We denote this segment by $K_C^*$. On this segment we define $(\lambda_0, \lambda)$ by means of (4.3), and apply the analysis of Sec. 4, except for the determination of initial data for (4.1). We now use the continuity of (5.8) at the junction point to determine $\lambda_0 C$ and $\lambda_C(\tau'')$, where the subscript $C$ refers to functions along $K_C^*$. Since we have $(\lambda_0, \lambda_B^-) \neq (0, \rho \theta \theta_x)$, it follows that at $(\tau'', x) \tau''$), the following jump condition holds:
\[ \lambda_C^- = \lambda_B^+ + \nu_C \theta_x \neq 0, \quad \nu_C \text{ arbitrary.} \]

Hence the conclusions of Theorem 2 of [1] hold along \( K_C^* \).

Let us now suppose that \( K_B^* \), instead of lying on the boundary as assumed in Sec. 5, is an interior segment. The point \((\tau, x(\tau))\), however, is still assumed to be a junction point. Along \( K_B^* \) we now define \((\lambda_0B^*, \lambda_B^*)\) by (4.3), and we apply the analysis of Sec. 4, except for the determination of the initial data for \((\lambda_0B^*, \lambda_B^*)\).

To determine \( \lambda_0B \) and \( \lambda_B(\tau) \) we use the continuity of (5.8) and get

\[ \lambda_B^- = \lambda_A^+ + (\nu_B^- - \nu_A^+ \theta_x). \]

Since we are free in our choice of \( \nu_B^- \), we get

(6.1) \[ \lambda_B^- = \lambda_A^+ + k \theta_x \neq 0. \]

Hence Theorem 2 of [1] holds along \( K_B^* \).

Note that in obtaining (6.1), we did not make use of the special fact that \( \nu_A^+ \) is arbitrary because the segment \( K_A^* \) terminates at \((t_1, x_1)\). Moreover, we can choose \( \nu_B^- \) so that \( k \geq 0 \).

We summarize the principal results of this paper in the following theorem.
Theorem 2. Let $u^* \in U$ be an optimal control, and let $K^*$ be the corresponding curve. Then there exists a constant $\lambda \geq 0$, an $n$-dimensional vector $\lambda(t)$, an $r$-dimensional vector $\mu(t) \leq 0$, and a function $\nu(t)$ such that the following hold: Along a segment of $K^*$ whose end points are junction points and that is in the interior of $\mathcal{B}$, except for the end-points, Theorem 2 of [1] holds. Along a segment whose end-points are junction points and that lies in $\mathcal{B}^*$, Theorem 1 of this paper holds. At a junction point either

\begin{equation}
(6.2) \quad \lambda_0 = 0, \quad \lambda^- + k \theta_x = 0, \quad k \neq 0,
\end{equation}

or

\begin{equation}
(6.3) \quad \lambda^+ = \lambda^- + k \theta_x \neq 0.
\end{equation}

At a junction point between two interior segments, (6.2) does not occur, and we may take $k > 0$ in (6.3). If (6.3) holds, then the following also holds:

\begin{equation}
(6.4) \quad H^+ = H^- - k \theta_t.
\end{equation}

From this theorem, several observations can be made. Since these are given by Gamkrelidze [4], there is no need to repeat them here.
REFERENCES


