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ANALYSIS OF MISSILE LAUNCHERS
Part J Phase 1
(Four-Degree-of-Freedom Multiple Launcher)

by
S. K. Lee
E. V. Wilms

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ANALYSIS OF MISSILE LAUNCHERS

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and
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A Research Project of the
Department of Theoretical and Applied Mechanics
University of Illinois

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This is the Jth part of the first phase report on University of Illinois Project No. 46-22-60-304, "Launcher Dynamics Study". The analysis was performed under contract No. DA-11-070-508-ORD-593. This particular report is an investigation of a four-degree of freedom multiple launcher including blast force effects.

The program is under the technical supervision of Rock Island Arsenal (RIA), Rock Island, Illinois, and the administrative supervision of Chicago Ordnance District.

Respectfully submitted
University of Illinois

[Signature]

M. Stippes, Project Supervisor
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LAUNCHER DYNAMICS STUDY

Part J Phase 1

Four-Degree-of-Freedom Multiple Launcher

ABSTRACT:

This report contains an analysis of a four degree of freedom model multiple launching system. The launcher pivots about a point and the motion is resisted by torsional springs. All of the missiles are parallel to one another in the launcher and are to be fired one by one in a pre-arranged order. The equations of motion of the system including the effects of blast are then established. In the second part, the effect of the blast force of the missile on the launcher face is considered.
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List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>(A, B, C)</td>
<td>reference system fixed to the earth</td>
</tr>
<tr>
<td>(X, Y, Z)</td>
<td>coordinates fixed in space whose Euler angles relative to (A, B, C) are</td>
</tr>
<tr>
<td></td>
<td>(γ_e, γ_a, 0)</td>
</tr>
<tr>
<td>(x, y, z)</td>
<td>coordinates fixed to the box at the pivot point O</td>
</tr>
<tr>
<td>(x'_m, y'_m, z'_m)</td>
<td>coordinates fixed to the mass center of the moving missile.</td>
</tr>
<tr>
<td>η</td>
<td>distance from mass center of the moving missile to xz-plane</td>
</tr>
<tr>
<td>γ_e</td>
<td>angle of elevation</td>
</tr>
<tr>
<td>γ_a</td>
<td>traverse angle of the launcher</td>
</tr>
<tr>
<td>(θ_1, θ_2, θ_3)</td>
<td>rotation of (x, y, z) relative to (X, Y, Z)</td>
</tr>
<tr>
<td>(o_o, y'_b, z'_b)</td>
<td>coordinates of O_b, the mass center of the launcher, in (x, y, z,)</td>
</tr>
<tr>
<td>(x'_m, y'_m, z'_m)</td>
<td>coordinates of O_M, mass center of the moving missile, in (x, y, z)</td>
</tr>
<tr>
<td>(x_i, y_o, z_i)</td>
<td>coordinates of O_I, mass center of the unfired missile in (x, y, z)</td>
</tr>
<tr>
<td>M_b, M_m</td>
<td>mass of the box and the missile respectively</td>
</tr>
<tr>
<td>M'_m</td>
<td>mass of the unfired missile M_m = M'_m in numerical value</td>
</tr>
<tr>
<td>I_{11}, I_{12}, etc.</td>
<td>mass moments of inertia of the launcher with respect to (x, y, z)</td>
</tr>
<tr>
<td>J_{11}, J_{12}, etc.</td>
<td>mass moments of inertia of the firing missile with respect to (x'_m, y'_m, z'_m)</td>
</tr>
<tr>
<td>J'<em>{11}, J'</em>{12}, etc.</td>
<td>mass moments of inertia of the unfired missile in the launcher with respect to (x'_m, y'_m, z'_m)</td>
</tr>
<tr>
<td>W_b, W_m</td>
<td>weight of the box and the missile respectively</td>
</tr>
<tr>
<td>W'_m</td>
<td>weight of the unfired missile W_m = W'_m in numerical value</td>
</tr>
<tr>
<td>i</td>
<td>the reverse of the order of firing i.e. the number of missiles left in the launcher</td>
</tr>
<tr>
<td>F(t)</td>
<td>thrust force of the missile</td>
</tr>
<tr>
<td>𝜋</td>
<td>dissipation function</td>
</tr>
</tbody>
</table>
\( T \) blast force on the launcher

\( \beta_{11} , \beta_{12} \) etc. torsional spring constants

\( \xi_{11} , \xi_{12} \) etc. damping coefficients

\( (X_T, \eta_b, Z_T) \) point of application of the blast force in \((x, y, z)\). \( \eta_b \) is the constant thickness of the launcher along y-axis. \( X_T, Z_T \) change with time when the missile leaves the launcher
Chapter 1

INTRODUCTION

This launching system differs from the others because it contains a number of missiles which can be fired successively in a launcher of box type. Referring to Fig. 1, we notice that the launcher is fixed at O where motion is resisted by torsional springs and dash pots. The launching tubes are parallel and are open at both ends. We also assume that there is no interference between exhaust cones of successive firings. In other words, at each time we consider the effect of one missile only.

The following information can be obtained from this report:

1. Equations of motion of the system before and after the missile leaves the launcher.
2. The linear displacement, velocity and acceleration of the mass center of the missile at end of guidance.
3. The displacements (angular) of the launcher.
4. An expression for computing the magnitude and point of application of the blast force acting on a part of a ring section.
Chapter II

MATHEMATICAL DESCRIPTION OF THE PROBLEM

1. Coordinate System:

(A, B, C) A fixed reference system on the earth

(X, Y, Z) Another fixed reference system on the earth. Y is parallel to the initial direction of the launcher tubes. This set of axes is obtained by rotating (A, B, C) through a traverse angle $\gamma_a$ about the C-axis, followed by a rotation of $\gamma_c$ about the X-axis (the new position of the A-axis after the rotation $\gamma_a$).

(x, y, z) This set of axes is fixed in the launcher at O and moves with it. The set is coincident with (X, Y, Z) initially.

(x_m, y_m, z_m) A set of axes fixed at the mass center of the missile which is being fired. These axes move with the missile.

(x'_m, y'_m, z'_m) Set of axes fixed at the mass center of an arbitrary unfired missile. These axes are always parallel to (x, y, z).

$\eta$ Since the missile always moves parallel to the tube, one coordinate is enough to describe completely its motion relative to the launcher. $\eta$ is measured from the mass center of the missile to the xz-plane.

2. Generalized Coordinates

This system has four degrees of freedom.

The orientation of the launcher may be described by three successive rotations $\theta_1, \theta_2, \theta_3$ as shown in Fig. 1. The position of a moving missile with respect to the
4. launcher is expressed by $\eta$. Thus the generalized coordinates are $\theta_1$, $\theta_2$, $\theta_3$, $\eta$.

3. Transformation of Coordinates

Following the same line of reasoning in TAM Report 139, p. 16, we obtain the transformation matrices

$$
\begin{bmatrix}
A \\ B \\ C
\end{bmatrix} = \begin{bmatrix} S \\ \mathbf{x} \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} \quad (1a)
$$

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2a)
$$

where

$$
[S] = \begin{bmatrix}
\ell_1 & -m_1 l_2 & m_1 m_2 \\
m_1 & \ell_1 l_2 & -l_1 m_2 \\
0 & m_2 & l_2
\end{bmatrix} \quad (1b)
$$

$$
[S][T] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \quad (3)
$$

$$
[T] = \begin{bmatrix}
1 - \frac{\theta_2}{2} & -\frac{\theta_3}{2} & \theta_1 \\
\theta_3 + \theta_1 \theta_2 & 1 - \frac{\theta_2}{2} & -\frac{\theta_3}{2} \\
-\theta_2 + \theta_1 \theta_3 & \theta_1 + \theta_2 \theta_3 & 1 - \frac{\theta_2}{2} - \frac{\theta_3}{2}
\end{bmatrix} \quad (2b)
$$

$m_1 = \sin \gamma_a$, $\ell_1 = \cos \gamma_a$

$m_2 = \sin \gamma_e$, $\ell_2 = \cos \gamma_e$
where

\[ a_{11} = l_1 \left( 1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} \right) - m_1 l_2 \left( \theta_3 + \theta_1 \theta_2 \right) + m_1 m_2 \left( -\theta_2 + \theta_1 \theta_3 \right) \]

\[ a_{12} = -l_1 \theta_3 - m_1 l_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + m_1 m_2 \left( \theta_1 + \theta_2 \theta_3 \right) \]

\[ a_{13} = l_1 \theta_2 + m_1 l_2 \theta_1 + m_1 m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \]

\[ a_{21} = m_1 \left( 1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} \right) + l_1 l_2 \left( \theta_3 + \theta_1 \theta_2 \right) - l_1 m_2 \left( -\theta_2 + \theta_1 \theta_3 \right) \]

\[ a_{22} = -m_1 \theta_3 + l_1 l_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) - l_1 m_2 \left( \theta_1 + \theta_2 \theta_3 \right) \]

\[ a_{23} = m_1 \theta_2 - l_1 l_2 \theta_1 - l_1 m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \]

\[ a_{31} = m_2 \left( \theta_3 + \theta_1 \theta_2 \right) + l_2 \left( -\theta_2 + \theta_1 \theta_3 \right) \]

\[ a_{32} = m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + l_2 \left( \theta_1 + \theta_2 \theta_3 \right) \]

\[ a_{33} = -m_2 \theta_1 + l_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \]

4. Position and Velocity Relations

\[ \overline{O_B} : \text{mass center of the launcher} \]

From Eqs. (1a), (2a), we obtain \( \overline{O \cdot O_B} \) in (A, B, C) as

\[
\begin{bmatrix}
A_b \\
B_b \\
C_b
\end{bmatrix} = [T] \begin{bmatrix}
S
\end{bmatrix} \begin{bmatrix}
y_b \\
z_b
\end{bmatrix}
\]
Then
\[ C_b = Y_b \left[ m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \]
\[ + Z_b \left[ -m_2 \theta_1 + \ell_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \]

(4)

\[ \text{O}_M: \text{ mass center of the firing missile} \]

Following the same procedure and taking the velocity components in (X, Y, Z) we obtain

\[
\begin{bmatrix}
V_{mx} \\
V_{my} \\
V_{mz}
\end{bmatrix} = \begin{bmatrix} \dot{X}_m \\ \eta \\ \dot{Z}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\eta} \\ 0 \end{bmatrix}
\]

(5a)

\[
V_{mx} = -x_m (\theta_2 \dot{\theta}_2 + \theta_3 \dot{\theta}_3) - \eta \dot{\theta}_3 + z_m \dot{\theta}_2 - \dot{\eta} \theta_3
\]

\[
V_{my} = x_m (\dot{\theta}_3 + \theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2) - \eta (\theta_1 \dot{\theta}_1 + \theta_3 \dot{\theta}_3) - z_m \dot{\theta}_1 + \dot{\eta} \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right)
\]

\[
V_{mz} = x_m (-\dot{\theta}_2 + \theta_1 \dot{\theta}_3 + \theta_2 \dot{\theta}_3) + \eta (\dot{\theta}_1 + \dot{\theta}_2 \theta_3 + \dot{\theta}_3 \theta_2) - z_m (\theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2)
\]

\[ + \dot{\eta} (\theta_1 + \theta_2 \theta_3) \]

(5b)

\[ C_m = x_m \left[ m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3) \right] \]
\[ + \eta \left[ m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \]
\[ + z_m \left[ -m_2 \theta_1 + \ell_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \]

(6)
$O = \text{mass center of an arbitrary unfired missile.}$

We assume here that $i$ is the reverse of the order of firing. $i$ lies between 1 and $k$. The $(k+1)^{\text{st}}$ missile is the one being fired. For every given $i$, we can locate the missile by the use of $(x_i, \eta_0, z_i)$. Note here $\eta_0$ is the same for all unfired missiles.

\[
\begin{bmatrix}
V_{ix} \\
V_{iy} \\
V_{iz}
\end{bmatrix} =
\begin{bmatrix}
x_i \\
\eta_0 \\
z_i
\end{bmatrix}
\]  

(7a)

\[
V_{ix} = -x_i (\dot{\theta}_2 + \dot{\theta}_3) - \eta_0 (\dot{\theta}_1 + \dot{\theta}_3) - z_i \dot{\theta}_2
\]

\[
V_{iy} = x_i (\dot{\theta}_3 + \dot{\theta}_1 + \dot{\theta}_2) - \eta_0 (\dot{\theta}_1 + \dot{\theta}_3) - z_i \dot{\theta}_1
\]  

(7b)

\[
V_{iz} = x_i (-\dot{\theta}_2 + \dot{\theta}_1 + \dot{\theta}_3) + \eta_0 (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) - z_i (\dot{\theta}_1 + \dot{\theta}_2)
\]

\[
C_i = x_i \left[ m_2 (\dot{\theta}_3 + \dot{\theta}_2) + l_2 (-\dot{\theta}_2 + \dot{\theta}_3) \right]
+ \eta_0 \left[ m_2 \left( \frac{\dot{\theta}_2^2}{2} - \frac{\dot{\theta}_3^2}{2} \right) + l_2 \left( \dot{\theta}_1 + \dot{\theta}_3 \right) \right]
+ z_i \left[ -m_2 \dot{\theta}_1 + l_2 (1 - \frac{\dot{\theta}_2^2}{2} - \frac{\dot{\theta}_3^2}{2}) \right]
\]  

(8)

$i$ ranges from 1 to $k$

\[F(t): \text{thrust force}\]

We assume here that the thrust force acts on $O_M$ and along the longitudinal axis of the missile
From
\[
\begin{bmatrix}
X_m \\
Y_m \\
Z_m
\end{bmatrix} = [T]
\begin{bmatrix}
x_m \\
\eta \\
z_m
\end{bmatrix}
\] (9a)

we obtain
\[
X_m = x_m \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2}\right) - \eta \theta_3 + z_m \theta_2
\]
\[
Y_m = x_m \left(\theta_3 + \theta_1 \theta_2\right) + \eta \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2}\right) - z_m \theta_1
\] (9b)
\[
Z_m = x_m \left(-\theta_2 + \theta_1 \theta_3\right) + \eta \left(\theta_1 + \theta_2 \theta_3\right) + z_m \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2}\right)
\]

The components of \( F \) are
\[
F_x = -F \theta_3
\]
\[
F_y = F
\]
\[
F_z = F \theta_1
\] (10)

5. Kinetic and Potential Energy of the System

\[
T = T_b + \sum T_m + T_m
\] (11)

where \( T_b, T_m, \sum T_m \) denote the kinetic energy of the launcher, of the firing missile and of the unfired missiles respectively.

\[
T_b = \frac{1}{2} \left(I_{11} \dot{\theta}_1^2 + I_{22} \dot{\theta}_2^2 + I_{33} \dot{\theta}_3^2 - 2I_{23} \dot{\theta}_2 \dot{\theta}_3\right)
\] (11a)

\( I_{12}, I_{13}, I_{23} \) vanish because of the symmetry of the launcher

\[
T_m = \frac{1}{2} \left[M_m \left(V_{mx}^2 + V_{my}^2 + V_{mz}^2\right) + J_{11} \dot{\theta}_1^2 + J_{22} \dot{\theta}_2^2 + J_{33} \dot{\theta}_3^2\right]
\] (11b)
\[ J_{12} = J_{13} = J_{23} = 0 \] due to symmetry of the missile.

\[ \sum T'_m = \sum_{i=1}^{K} \frac{1}{2} \left[ M'_m (V'_{ix} + V'_{iy} + V'_{iz}) + J'_{11} \dot{\theta}_1^2 + J'_{22} \dot{\theta}_2^2 + J'_{33} \dot{\theta}_3^2 \right] \] (11c)

Potential Energy

\[ V = V_b + \sum V'_m + V_m + V_s \] (12)

where \( V_b, V'_m, \sum V'_m, V_s \) denote the potential energy of the launcher, of the firing missile, of the unfired missiles, and of the spring, respectively.

\[ V_b = W_b C_b \]
\[ = W_b \left\{ y_b \left[ m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \right. \]
\[ \left. + z_b \left[ -m_2 \theta_1 + \ell_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \right\} \] (12a)

\[ V_m = W_m C_m \]
\[ = W_m \left\{ x_m \left[ m_2 (\theta_3 + \theta_1 \theta_2) + \ell_2 (-\theta_2 + \theta_1 \theta_3) \right] \right. \]
\[ \left. + \eta \left[ m_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_3^2}{2} \right) + \ell_2 (\theta_1 + \theta_2 \theta_3) \right] \right. \]
\[ \left. + z_m \left[ -m_2 \theta_1 + \ell_2 \left( 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} \right) \right] \right\} \] (12b)
\[ \sum V'_{m} = \sum_{i=1}^{k} W'_{m} C_{i} \]

\[ = W'_{m} \sum_{i=1}^{k} \left\{ x_{i} \left[ m_{2} \left( \theta_{3} + \theta_{1} \theta_{2}^{*} \right) + \ell_{2} \left( -\theta_{2}^{*} + \theta_{1} \theta_{3}^{*} \right) \right] ight. \]

\[ + z_{1} \left[ -m_{2} \theta_{1}^{2} + \ell_{2} \left( 1 - \frac{\theta_{2}^{2}}{2} - \frac{\theta_{3}^{2}}{2} \right) \right] \left( 12c \right) \]

\[ + W'_{m} k \eta_{0} \left[ m_{2} \left( 1 - \frac{\theta_{2}^{2}}{2} - \frac{\theta_{3}^{2}}{2} \right) + \ell_{2} \left( \theta_{1} + \theta_{2} \theta_{3} \right) \right] \]

\[ V_{s} = \frac{1}{2} \left( \beta_{11} \theta_{1}^{2} + \beta_{22} \theta_{2}^{2} + \beta_{33} \theta_{3}^{2} + 2 \beta_{12} \theta_{1} \theta_{2} \right) \]

\[ + \quad 2 \beta_{13} \theta_{1} \theta_{3} + 2 \beta_{23} \theta_{2} \theta_{3} \quad (12d) \]

Dissipation due to dash pots

\[ \mathcal{F} = \frac{1}{2} \left( c_{11} \dot{\theta}_{1}^{2} + c_{22} \dot{\theta}_{2}^{2} + c_{33} \dot{\theta}_{3}^{2} + 2 c_{12} \dot{\theta}_{1} \dot{\theta}_{2} \right) \]

\[ + \quad 2 c_{13} \dot{\theta}_{1} \dot{\theta}_{3} + 2 c_{23} \dot{\theta}_{2} \dot{\theta}_{3} \quad (13) \]

6. Equations of Motion

If \( \theta_{i} \quad (i = 1, 2, 3, 4) \) denotes the generalized coordinates \( (\theta_{1}, \theta_{2}, \theta_{3}, \eta) \),

Lagrange's equations of motion are:

\[ \frac{d}{dt} \left[ \frac{\partial T}{\partial \dot{\theta}_{i}} \right] - \frac{\partial T}{\partial \theta_{i}} = \frac{\partial V}{\partial \theta_{i}} - F_{x} \frac{\partial X}{\partial \theta_{i}} + F_{y} \frac{\partial Y}{\partial \theta_{i}} + F_{z} \frac{\partial Z}{\partial \theta_{i}} \]

\[ + \quad \frac{\partial Z}{\partial \theta_{i}} \quad (14) \]

The above equation yields four differential equations. Three of these may be written in indicial form as follows:

\[ a_{i j} \ddot{\theta}_{j} + b_{i j} \dot{\theta}_{j} + c_{i j} \theta_{j} + d_{i} = 0 \quad (15) \]

\[ (i, j = 1, 2, 3) \]
where

\[ a_{11} = I_{11} + M_m (z_m^2 + \eta_m^2) + J_{11} + M_m' \sum_{i=1}^{\kappa} (z_i^2 + \eta_i^2) + k J'_{11} \]

\[ a_{12} = -M_m x_m \eta - M_m' \eta_0 \sum_{i=1}^{\kappa} x_i \]

\[ a_{13} = -M_m x_m z_m - M_m' \sum_{i=1}^{\kappa} x_i z_i \]

\[ a_{21} = -M_m x_m \eta - M_m' \eta_0 \sum_{i=1}^{\kappa} x_i \]

\[ a_{22} = I_{22} + M_m (z_m^2 + x_m^2) + J_{22} + M_m' \sum_{i=1}^{\kappa} (x_i^2 + z_i^2) + k J'_{22} \]

\[ a_{23} = -I_{23} - M_m z_m \eta - M_m' \sum_{i=1}^{\kappa} z_i \eta_0 \]

\[ a_{31} = -M_m x_m z_m - M_m' \sum_{i=1}^{\kappa} x_i z_i \]

\[ a_{32} = -I_{23} - M_m z_m \eta - M_m' \sum_{i=1}^{\kappa} z_i \eta_0 \]

\[ a_{33} = I_{33} + M_m (\eta_m^2 + x_m^2) + J_{33} + M_m' \sum_{i=1}^{\kappa} (\eta_i^2 + x_i^2) + k J'_{33} \]

\[ b_{11} = 2 M_m \eta \dot{\eta} + c_{11} \]

\[ b_{12} = c_{12} \]

\[ b_{13} = c_{13} \]

\[ b_{21} = -2 M_m x_m \dot{\eta} + c_{12} \]

\[ b_{22} = c_{22} \]

\[ b_{23} = -2 M_m z_m \dot{\eta} + c_{13} \]

\[ b_{31} = c_{13} \]
\[ b_{32} = c_{23} \]

\[ b_{33} = 2 M_m \eta \ddot{\eta} + c_{33} \]

\[ c_{11} = -W_b (y_b m_{2} + z_b \ell_{2}) - W_m (\eta m_{2} + z_m \ell_{2}) - W'_m \sum_{i=1}^{N} x_i m_{2} + \beta_{11} \]

\[ - W'_m k \eta_0 m_{2} + \beta_{11} \]

\[ c_{12} = M_m x_m \ddot{\eta} + W_m x_m m_{2} + W'_m \sum_{i=1}^{N} x_i m_{2} + \beta_{12} - F x_m \]

\[ c_{13} = W_m x_m \ell_{2} + W'_m \sum_{i=1}^{N} x_i \ell_{2} + \beta_{13} \]

\[ c_{21} = W_m x_m m_{2} + W'_m \sum_{i=1}^{N} x_i + \beta_{12} \]

\[ c_{22} = -W_b z_b \ell_{2} - W_m z_m \ell_{2} - W'_m \sum_{i=1}^{N} z_i \ell_{2} + \beta_{22} \]

\[ c_{23} = W_b y_b \ell_{2} + W_m \eta \ell_{2} + W'_m k \eta_0 \ell_{2} + \beta_{23} - M_m z_m \ddot{\eta} + F z_m \]

\[ c_{31} = W_m x_m \ell_{2} + W'_m \sum_{i=1}^{N} x_i \ell_{2} + \beta_{13} \]

\[ c_{32} = W_b \ell_{2} + W_m \eta \ell_{2} + W'_m k \eta_0 \ell_{2} + \beta_{23} \]

\[ c_{33} = -W_b y_b m_{2} - W_m m_{2} \eta - W'_m k \eta_0 m_{2} + \beta_{33} \]

\[ d_1 = -M_m z_m \ddot{\eta} + W_b (y_b \ell_{2} - z_b m_{2}) + W_m (\eta \ell_{2} - z_m m_{2}) - W'_m \sum_{i=1}^{N} z_i m_{2} + W'_m k \eta_0 \ell_{2} + F z_m \]

\[ d_2 = -W_m x_m \ell_{2} - W'_m \sum_{i=1}^{N} x_i \ell_{2} \]

\[ d_3 = M_m x_m \ddot{\eta} + W_m x_m m_{2} + W'_m \sum_{i=1}^{N} x_i m_{2} - F x_m \]

The fourth equation is

\[-M_m z_m \ddot{\eta} + M_m x_m \ddot{\eta} + W_m \ell_{2} \theta + M_m \ddot{\eta} + W_m m_{2} - F = 0 \quad (16)\]
7. Free Motion of the Launcher

After the missile leaves the launcher, the latter will assume free motion. Eq. (16) becomes trivial and vanishes identically. The required equations of motion can be obtained by putting

\[ \eta = \eta_0, \quad M_m = 0, \quad W_m = 0, \quad J_{11} = J_{12} \ldots = 0 \]

\[ x_m = X_T, \quad z_m = Z_T, \quad F = -T \]

The last term, \( F = -T \), is due to the exhaust of the missile leaving the launcher. Also note \( X_T, \quad Z_T \) are quite different from \( x_m, \quad z_m \).

The equations are

\[ a'_{ij} \ddot{\theta}_j + b'_{ij} \ddot{\theta}_j + c'_{ij} \dot{\theta}_j + d'_i = 0 \quad (17) \]

\( i, \quad j = 1, \quad 2, \quad 3 \)

where

\[ a'_{11} = I_{11} + M'_m \sum_{i=1}^{k} \left( z_i^2 + \eta_0^2 \right) + k_j'_{11} \]

\[ a'_{12} = -M'_m \eta_0 \sum_{i=1}^{k} x_i \]

\[ a'_{13} = -M'_m \sum_{i=1}^{k} x_i z_i \]

\[ a'_{21} = -M'_m \eta_0 \sum_{i=1}^{k} x_i \]

\[ a'_{22} = I_{22} + M'_m \sum_{i=1}^{k} \left( x_i^2 + z_i^2 \right) + k_j'_{22} \]

\[ a'_{23} = -I_{23} - M'_m \sum_{i=1}^{k} z_i \eta_0 \]

\[ a'_{31} = -M'_m \sum_{i=1}^{k} x_i z_i \]

\[ a'_{32} = -I_{32} - M'_m \sum_{i=1}^{k} x_i \eta_0 \]

\[ a'_{33} = I_{33} + M'_m \sum_{i=1}^{k} \left( x_i^2 + z_i^2 \right) + k_j'_{33} \]
\[ a'_{32} = - I_{23} - M' m \eta_o \sum_{i=1}^{k} z_i \]
\[ a'_{33} = I_{33} + M' m \sum_{i=1}^{k} (\eta_o^2 + x_1^2) + k J'_{33} \]
\[ b'_{11} = c_{11} \]
\[ b'_{12} = c_{12} \]
\[ b'_{13} = c_{13} \]
\[ b'_{21} = c_{12} \]
\[ b'_{22} = c_{22} \]
\[ b'_{23} = c_{23} \]
\[ b'_{31} = c_{13} \]
\[ b'_{32} = c_{23} \]
\[ b'_{33} = c_{33} \]
\[ c'_{11} = - W_{b} (y_{b} m_2 + z_{b} \ell_2) - W'_{m} \sum_{i=1}^{k} z_i \ell_2 - W'_{m} k \eta_o m_2 + \beta_{11} \]
\[ c'_{12} = W'_{m} \sum_{i=1}^{k} x_1 m_2 + \beta_{12} + T X_T \]
\[ c'_{13} = W'_{m} \sum_{i=1}^{k} x_1 \ell_2 + \beta_{13} \]
\[ c'_{21} = W'_{m} m_2 \sum_{i=1}^{k} x_1 + \beta_{12} \]
\[ c'_{22} = - W_{b} z_{b} \ell_2 - W'_{m} \sum_{i=1}^{k} z_i \ell_2 + \beta_{22} \]
\[ c'_{23} = W_{b} y_{b} \ell_2 + W'_{m} k \eta_o \ell_2 + \beta_{23} - T Z_T \]
\[ c'_{31} = W' m \sum_{i=1}^{k} x_i \ell_2 + \beta_{13} \]
\[ c'_{32} = W' b \ell_2 + W' m k \eta_o \ell_2 + \beta_{23} \]
\[ c'_{33} = -W' b y_b m_2 - W' m k \eta_o m_2 + \beta_{33} \]
\[ d'_1 = W' (y_b \ell_2 - z_b m_2) - W' m \sum_{i=1}^{k} z_i m_2 + W' m k \eta_o \ell - T Z_T \]
\[ d'_2 = -W' m \sum_{i=1}^{k} x_i \ell_2 \]
\[ d'_3 = W' m \sum_{i=1}^{k} x_i m_2 + T X_T \]
Chapter III

COMPUTATION OF THE BLAST FORCE

To calculate the blast force acting on the launcher due to the exhaust field of the rocket, we use an approximate value for the stagnation pressure the coordinates of which are fixed in the rocket. This force will be incorrect due to the velocity of the rocket; however, since we are also neglecting interaction between the exhaust jet and the launcher, we will consider it sufficiently accurate to adopt this point of view. The calculation is based on the method outlined in the following reference:


The following notation will be used in this section:

- $S$ the distance measured from the nozzle exit of the missile along its axis of symmetry
- $r$ the distance from a point to the axis of symmetry of the missile
- $p(S, r)$ total gage pressure at point $(S, r)$
- $p_m$ total gage pressure at the axis of symmetry
- $p_e$ pressure at the nozzle exit

\[ p_e = p_c \left[ \frac{\gamma + 1}{2} \left( \frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right)^{\gamma - 1} \right] \left[ \frac{2 \gamma}{\gamma + 1} M^2 - \frac{\gamma - 1}{\gamma + 1} \right]^{\frac{1}{1 - \gamma}} - p_a \]

- $p_c$ rocket motor chamber pressure
- $\gamma$ ratio of specific heats
Mach number of gas flow at exit of nozzle. May be computed from:

\[
\frac{A_t}{A_e} = M \left[ \frac{\gamma + 1}{2 + (\gamma - 1) M^2} \right] \frac{\gamma + 1}{2(\gamma - 1)}
\]

- \(A_t\) area of nozzle at throat
- \(A_e\) area of nozzle at exit
- \(P_a\) atmospheric pressure
- \(S_o\) for \(S \leq S_o\), \(P_m = P_e\)
- \(d_e\) diameter of nozzle exit
- \(r_e\) radius of nozzle exit

\[
k_1 = P_a \left[ \left( \frac{\gamma + 1}{2} \right)^{\gamma - 1} - 1 \right] \left[ 16 (M - 1) d_e \right]^{-k_2}
\]
\[
k_2 = - (1.4 + 0.437 M)
\]
\[
k_3 = d_e (S/d_e)_1
\]

\((S/d_e)_1\) is given by Fig. 2

\(r_o\) the reference point where the pressure falls to 1/4 of its value on the axis. \(r_o\) is given by

\[
r_o = r_e \quad \text{if} \quad \frac{S/d_e}{(S/d_e)_1} < 1
\]
\[
r_o = r_e \left[ \frac{S/d_e}{(S/d_e)_1} \right]^{1.16} \quad \text{if} \quad \frac{S/d_e}{(S/d_e)_1} \geq 1
\]

\(x_s, z_s\) rectangular coordinates in the plane (surface of the launcher) normal to \(S\) at the center of the hole parallel to the z-axis

\(R_o, R_i\) outer and inner radius of a given ring section about the center of the hole
CROSSPLOT OF SPREADING CHARACTERISTICS
VERSUS MACH NUMBER

\[ 0.1 = \frac{a_1}{a_L} \sqrt{\frac{\gamma}{\gamma - 1}} (\frac{\sigma}{\rho / x}) \]
\( \phi_1, \phi_2 \) angles measured with respect to \( x_s \); they are the circumferential locations of the ring section

\( T_1 \) the blast force on the above ring section

\( R_e \) effective radius of a ring section

\( M_x, M_z \) moments about \( x_s, z_s \) respectively, of the ring section

\( \text{Erf}(x) \) error function

\[
\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\lambda^2} d\lambda
\]
The pressure at any point \((S, r)\) is given by

\[
P(S, r) = a e^{-b^2 r^2}
\]

(18)

There are four cases for the above expression with different \(a, b\):

i. \(0 < S < S_o, \quad \frac{S}{k_3} < 1\)

\[
a = P_e
\]

\[
b^2 = \left( \frac{1.15}{r_e} \right) ^2
\]

ii. \(0 < S < S_o, \quad \frac{S}{k_3} \geq 1\)

\[
a = P_e
\]

\[
b^2 = \left( \frac{1.15}{r_e} \right) ^2 \left( k_3 \right) ^{2.32} \quad (S)^{-2.32}
\]

iii. \(S > S_o, \quad \frac{S}{k_3} < 1\)

\[
a = k_1 S^2
\]

\[
b^2 = \left( \frac{1.15}{r_e} \right) ^2
\]

iv. \(S > S_o, \quad \frac{S}{k_3} \geq 1\)

\[
a = k_1 S^2
\]

\[
b^2 = \left( \frac{1.15}{r_e} \right) ^2 \left( k_3 \right) ^{2.32} \quad (S)^{-2.32}
\]

Using Eq. (18) we obtain

\[
T_i = \int_{\Theta_1}^{2\Pi} \int_{R_i}^{R_o} P \, dA = \int_{\Theta_1}^{2\Pi} \int_{R_i}^{R_o} P \, r \, dr \, d\Phi
\]
or

\[ T_1 = \frac{a}{2b^2} \left( \phi_2 - \phi_1 \right) \left( e^{-b^2R_1^2} - e^{-b^2R_0^2} \right) \]  \hspace{1cm} (19)

or

\[ R_e = \frac{\int_{R_i}^{R_o} \rho \, dA}{\int_{R_i}^{R_o} \rho \, dA} \]

\[ R_e = \left\{ \frac{4\pi}{2b} \left[ \text{Erf} (bR_0) - \text{Erf} (bR_1) \right] - \left( R_o e^{-b^2R_0^2} - R_i e^{-b^2R_1^2} \right) \right\} \]

\[ dM_z = -R_e \cos \phi \int_{R_i}^{R_o} \rho \, dr \, d\phi \frac{2\pi}{2\pi} \]

\[ = -R_e \cos \phi \frac{d\phi}{2\pi} \int_{R_i}^{R_o} \rho \, 2\pi \, dr \]

\[ M_z = -\frac{T_1 R_e}{\phi_2 - \phi_1} \left( \sin \phi_2 - \sin \phi_1 \right) \] \hspace{1cm} (21)

Similarly

\[ M_x = \frac{T_1 R_e}{\phi_2 - \phi_1} \left( \cos \phi_2 - \cos \phi_1 \right) \] \hspace{1cm} (22)

From Eqs. (19), (21), (22), with all the required data, we can find the magnitude and point of application of the force acting on any given part of a ring section at a given time. Eqs. (15), (16), should yield the accelerations and velocities of the launcher and missile as the missile is on the point of leaving the launcher. \( S \) may be expressed as a function of time, i.e. \( S = S(t) \). This must be obtained from the rocket motion after firing.
At a given time \( t \), the effective area on the launcher face covered by the exhaust cone (use \( r = r_0 \) for its computation) is known. We divide this area into a number of suitably chosen ring sections with respect to the center of the hole for the particular rocket. The magnitude and point of application of the blast force on each ring section are computed; then we can find the magnitude and location of the resultant force. This will give \( T \) and \( X_T, Z_T \) in Eq. (17).

Note that in the calculation of the effective areas, the effect due to holes on the launcher face is neglected.
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