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A NOTE ON SEMIDEFINITE MATRICES

by

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ABSTRACT

It is of general interest to find criteria for a matrix to be positive (or negative)- semidefinite. The usual characterization of semidefinite matrices in terms of their principal minors can be rather laborious to implement practically. We present here an elementary proof of a known alternate characterization of a semidefinite matrix in terms of its null-space and of its largest characteristic value. An iterative procedure is also suggested which may be useful in deciding the semidefiniteness of a matrix.
A NOTE ON SEMIDEFINITE MATRICES

In what follows $A$ will always represent a real, symmetric, $n \times n$ matrix. If, for each $x \in \mathbb{R}^n$ (*) it is true that $x A x^T \geq 0$ (**) then we say that $A$ is positive-semidefinite, denoted: p.s.d.; if $(x A x^T)(y A y^T) > 0$ for all $x, y \in \mathbb{R}^n$ we say that $A$ is semidefinite, denoted s.d.. We first prove the following:

THEOREM 1. The following are equivalent:

(i) $A$ is s.d.

(ii) $(x A y^T)^2 \leq (x A x^T)(y A y^T)$, all $x, y \in \mathbb{R}^n$

(iii) $x \in \mathbb{R}^n$, $x A x^T = 0 \iff x A^2 x^T = 0$

(iv) $x \in \mathbb{R}^n$, $x A^2 x^T = 1 \Rightarrow (x A x^T)^2 > 0$

(v) $x \in \mathbb{R}^n$, $x A x^T = 0 \Rightarrow x A = 0$

PROOF: We show (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (i).

(i) $\implies$ (ii)

Suppose $A$ is s.d., let $x, y \in \mathbb{R}^n$. Consider the real quadratic polynomial $p$ defined by:

$$p(\lambda) = (x + \lambda y) A (x + \lambda y)^T =$$

$$= x A x^T + 2\lambda x A y^T + \lambda^2 y A y^T.$$

Since $A$ is s.d., $p$ does not change sign, i.e., its discriminant is non-positive, whence:

$$4(x A y^T)^2 - 4(x A x^T)(y A y^T) \leq 0,$$

(*)

$$\mathbb{R}^n = \left\{ x \mid x = (x_1, \ldots, x_n) \text{ and } x_i \text{ is a real number for } i = 1, \ldots, n \right\}.$$  

(**) If $x \in \mathbb{R}^n$, $x^T$ denotes the transpose of $x$. 


giving the desired result.

(ii) \(\Rightarrow (iii)\).

Suppose \(x \in \mathbb{R}^n\) and \(xAx^T = 0\), then, from (ii), \((xAy)^T \leq 0\), i.e.,
\[xAy^T = 0,\]
for all \(y \in \mathbb{R}^n\). Thus \(xA = 0\), but \(xA^2x^T = (xA)(xA)^T = 0\).

(iii) \(\Rightarrow (iv)\).

If \(x \in \mathbb{R}^n\) and \((xAx^T)^2 \leq 0\) then \(xAx^T = 0\) and, by (iii), \(xA^2x^T = 0\), contradicting \(xA^2x^T = 1\).

(iv) \(\Rightarrow (v)\).

If \(x \in \mathbb{R}^n\) and \(xAx^T = 0\) then, by (iv), \(xA^2x^T \leq 0\) (because if \(xA^2x^T > 0\) then we could normalize \(x\) to get \(xA^2x^T = 1, xAx^T = 0\)). However, \(xA^2x^T = (xA)(xA)^T\), and thus \(xA^2x^T \geq 0\) with equality holding if and only if \(xA = 0\).

(v) \(\Rightarrow (i)\).

Suppose (i) is false, i.e., there exist \(x, y \in \mathbb{R}^n\) such that \(xAx^T > 0\), \(yAy^T < 0\). By suitable normalization [dividing \(x\) by \((xAx)^{1/2}\) and \(y\) by \((-yAy)^{1/2}\)], we may assume that \(xAx^T = 1, yAy^T = -1\). Now let:

1. \[\lambda = -xAy^T + \left[1 + (xAy^T)^2\right]^{1/2}\]
2. \[z = \lambda x + y\]

We claim that \(zA \neq 0\) and \(zAz^T = 0\), thus contradicting (v). First, if \(zA = 0\) then multiplying (2) by \(Ax^T\) and \(Ay^T\) we get:

\[0 = \lambda xAx^T + yAx^T = \lambda + xAy^T\]
\[0 = \lambda xAy^T + yAy^T = \lambda xAy^T - 1\]

Combining the last two equations:

\[0 = \lambda xAy^T - 1 = (-xAy^T)(xAy^T) - 1 = -1 - (xAy^T)^2,\]
a contradiction, thus $zA \neq 0$. However,

$$zAz^T = (\lambda x + y)A(\lambda x + y)^T =$$

$$= \lambda^2 xAx^T + 2\lambda xAy^T + yAy^T$$

$$= \lambda^2 + 2\lambda xAy^T - 1,$$

and $\lambda$ was chosen to be precisely one of the two (real) roots of the preceding quadratic polynomial in $\lambda$. q.e.d.

Several comments are in order. Obviously, $A$ is s.d. if and only if $A$ is p.s.d. or $-A$ is p.s.d. Condition (ii) of Theorem 1 is a generalization of the Cauchy-Schwartz inequality, namely:

$$(3) \quad (uv^T)^2 \leq (uu^T)(vv^T) \quad \text{all } u, v \in \mathbb{R}^n,$$

for if we take $A$ to be the $n \times n$ identity matrix which is clearly p.s.d., we obtain (3) from (ii) - Theorem 1. Condition (v) - Theorem 1, or its obvious equivalents (iii) and (iv), states that if we consider $xA$, the image under the linear transformation $A$ of a point $x$ in $\mathbb{R}^n$, then $A$ cannot be perpendicular to $x$ unless $x$ is in the null-space of $A$. Alternately, (v) - Theorem 1 states that if $x$ is not in the null-space of $A$ then its image under $A$ cannot be perpendicular to $x$.

We proceed next to obtain results which are, in a sense, "refinements" of conditions (ii) (see Lemma 1 below) and (iv) (see Theorem 2) of Theorem 1. Lemma 1 is a generalization of the well known fact associated with the Cauchy-Schwartz inequality, stating that equality holds in (3) if and only if $u, v$ are linearly dependent. We shall apply Lemma 1 in the proof of Theorem 3.
LEMMA 1
Let $A$ be s.d. If $x, y \in \mathbb{R}^n$ then $(x^T A y)^2 = (x^T A x)(y^T A y)$ if and only if $x^A, y^A$ are linearly dependent.

PROOF: If, say, $x^A = \lambda y^A$, where $\lambda$ is a real number, then $x^T A y = \lambda y^T A y$ while $x^T A x = \lambda x^T A x = \lambda^2 y^T A y$. Whence it follows that $(x^T A y)^2 = \lambda^2 (y^T A y)^2 = (x^T A x)(y^T A y)$.

On the other hand, suppose $(x^T A y)^2 = (x^T A x)(y^T A y)$. If $x^T A x = 0$ or $y^T A y = 0$ then, by (v) - Theorem 1, $x^A = 0$ or $y^A = 0$ and we certainly can conclude that $x^A, y^A$ are linearly dependent. Otherwise, say, $x^T A x > 0$ and $y^T A y > 0$, consequently $x^T A y \neq 0$. Let $\rho = \text{signum} (x^T A y)$ and let:

$$\alpha = (y^T A y)^{1/2}$$
$$\beta = -\rho (x^T A x)^{1/2},$$

then $\alpha, \beta \neq 0$ and:

$$(x + \beta y)^T A (x + \beta y) = \alpha^2 x^T A x + \beta^2 y^T A y + 2\alpha \beta x^T A y =$$
$$= 2(x^T A x)(y^T A y) - 2\rho (x^T A x)(x^T A y)^{1/2}(y^T A y)^{1/2} =$$
$$= 2(x^T A y)^2 - 2\|x^T A y\|^2 =$$
$$= 2(x^T A x)(y^T A y) - 2(x^T A x)(y^T A y)^{1/2} = 0.$$ 

Thus, $(x + \beta y)^T A (x + \beta y) = 0$ and, by (v) - Theorem 1, $0 = (x + \beta y)^T A x = x^A + \beta y^A$. q.e.d.

The preceding lemma was motivated, in part, by an examination of (ii) - Theorem 1 in case $A$ is the identity matrix, in that case (since the square of the identity is the identity), (iv) - Theorem 1 states: $x \in \mathbb{R}^n$, $x^T x = 1$ implies $(x^T x)^2 > 0$, which is, of course, true. We notice, though, that $(x^T A x)^2$ has then a positive lower bound, namely 1. In general, this
will be the case, i.e., a positive lower bound will exist for \((xAx^T)^2\) in (iv) - Theorem 1, whenever \(A\) is s.d.. Clearly, when \(A\) is identically zero any positive number will serve as a lower bound, because there is no \(x \in \mathbb{R}^n\) for which \(xA^2x^T = 1\), thus we will exclude \(A = 0\) in the next theorem:

**THEOREM 2**

Suppose \(A\) is p.s.d. and \(A \neq 0\), then there exist a positive real number \(\mu\) and an \(x_0 \in \mathbb{R}^n\) such that:

\[
\begin{align*}
(4) & \quad x \in \mathbb{R}^n, xA^2x^T = 1 \Rightarrow xAx^T \geq \mu \\
(5) & \quad x_0A^2x_0^T = 1 \quad \text{and} \quad x_0Ax_0^T = \mu.
\end{align*}
\]

**PROOF:** Let

\[
X = \left\{ x \mid x \in \mathbb{R}^n \quad \text{and} \quad xA^2x^T = 1 \right\}
\]

\[
\mu = \inf_{x \in X} xAx^T.
\]

Since \(A\) is p.s.d. and \(A \neq 0\), \(\mu\) is well defined and in fact \(\mu \geq 0\) and satisfies (4). By definition of \(\mu\), there exists a sequence \(x_k\) such that

\[
\begin{align*}
(6) & \quad x_k \in X \quad \text{for} \quad k = 1, 2, \ldots \\
(7) & \quad x_kAx_k^T \quad \text{converges to} \quad \mu.
\end{align*}
\]

We consider two cases:

**Case 1.** The sequence \(x_k\) has a bounded subsequence. In this eventuality the \(x_k\) have a point of accumulation \(x_0\), for which it must be true (by (6) and (7) and because \(X\) is closed) that \(x_0 \in X\) and \(x_0Ax_0^T = \mu\). Thus \(x_0\) satisfies (5). That \(\mu\) is positive then follows from (v) - Theorem 1. The two preceding facts, together with the remark above that \(\mu\) satisfies 4, complete the proof.
Case 2. The sequence \( \{x_k\} \) has no bounded subsequence, i.e., we may assume that 
\[ |x_k| = (x_k^T x_k)^{1/2} \to \infty \quad \text{and} \quad |x_k| > 0, \quad k = 1, 2, \ldots \]
We define another sequence \( \{y_k\} \) by:
\[
y_k = \frac{x_k}{|x_k|}
\]
Now, \( y_k Ax_k^T \) converge to zero, because \( x_k Ax_k^T \) converge to \( \mu \) and also \( y_k A^2 y_k^T \) converge to zero, because \( x_k A^2 x_k = 1 \) all \( k \). However, \( |y_k| = 1 \), thus the \( y_k \)'s have an accumulation point \( y \), for which it must be true that \( yA y^T = 0 \). Thus \( yA = 0 \) by (v) - Theorem 1.

Next we observe that from the definition of \( y \) and the \( y_k \)'s it follows that whenever \( y \) has a non-zero component then infinitely many \( x_k \)'s have the same component non-zero, and in fact of the same sign. We may assume that an appropriate subsequence of \( x_k \) has been selected so that whenever \( y \) has a positive (negative) component then all the \( x_k \)'s have the same component positive (negative). Now, if \( \{\lambda_k\} \) is any sequence of real numbers then:
\[
(x_k + \lambda_k y)x_k^T = x_k Ax_k^T
\]
and
\[
(x_k + \lambda_k y)A^2 x_k^T = x_k A^2 x_k^T,
\]
because \( yA = 0 \). We can thus replace \( x_k \) by \( x_k + \lambda_k y, \quad k = 1, 2, \ldots \), and (6) and (7) will still hold. However, by an appropriate choice of \( \lambda_k \) we can reduce the number of non-zero components in each of the \( x_k \)'s, eventually (repeating the above process, if necessary) we obtain a sequence \( \{x_k\} \), satisfying (6)-(7) and which has an accumulation point, thus reducing it to case 1.

q.e.d.
As an immediate consequence of Theorem 2 we can "strengthen" (iv) - Theorem 1.

**Corollary**

If \( A \) is s.d. and \( A \neq 0 \) then

\[
\text{minimum } \left\{ \left( xAx^T \right)^2 \mid x \in \mathbb{R}^n \text{ and } xA^2x^T = 1 \right\}
\]

exists and is positive.

**PROOF:** As noted before, if \( A \) is s.d., then either \( A \) is p.s.d. or \(-A\) is p.s.d., in either case the square of the \( \mu \) in Theorem 2 is the required minimum and the \( x_0 \) of the same theorem is the required minimizing \( x \).

The \( \mu \) and \( x_0 \) of Theorem 2 are, as one might expect, intimately related to the characteristic values of \( A \). This is brought forth in the next theorem.

**THEOREM 3**

Let \( A \) be p.s.d., \( A \neq 0 \). Let \( \mu \) and \( x_0 \) be as in Theorem 2 and let \( \lambda_n \) be the largest characteristic value of \( A \), then \( \lambda_n = \mu^{-1} \) and \( x_0A \) is a characteristic vector of \( A \) corresponding to \( \lambda_n \).

**PROOF:** Suppose \( \lambda \) is any characteristic value of \( A \), i.e., there exists an \( x \in \mathbb{R}^n, x \neq 0 \), such that \( xA = \lambda x \), whence \( xA^2x^T = \lambda xAx^T \). If \( \lambda = 0 \) then certainly \( \lambda \leq \mu^{-1} \). Assuming \( \lambda \neq 0 \), it follows that \( xA \neq 0 \) (because \( x \neq 0 \)) and thus, by (v) - Theorem 1, \( xAx^T > 0 \). Let \( y = (xA^2x^T)^{-1/2}x \), then \( yA^2y = 1 \) and, by definition of \( \mu \), \( yAy^T \geq \mu \). However \( yAy^T = (xA^2x^T)^{-1}(xAx^T) = \lambda^{-1} \), thus \( \lambda \leq \mu^{-1} \). We have just demonstrated that \( \lambda \leq \mu^{-1} \) for any characteristic value \( \lambda \) of \( A \), thus \( \lambda_n \leq \mu^{-1} \).
To complete the proof of this theorem it will suffice to show that there is a characteristic value $\lambda$ of $A$ such that $\lambda = \mu^{-1}$, and $(x_0 A) A = \lambda(x_0 A)$, $x_0$ being as in Theorem 2. Let $x = x_0$ be a minimizing $x_0$ in question. Since $A$ and $A^2$ are p.s.d. (the square of any real symmetric matrix is p.s.d.), and $x A \neq 0 (x A x^T = x_0 A x_0^T = \mu > 0)$, it follows that $x A^3 x^T = (x A) A (x A)^T > 0$, and $x A^4 x = (x A) A^2 (x A)^T > 0$. Thus, if we define

\[(8) \quad \rho = 2(x A^3 x^T)(x A^4 x^T)^{-1}\]

then $\rho$ is positive. Next let

\[(9) \quad y = x - \rho x A .\]

The motivation for the above definition of $y$ is as follows: we know $x$ minimizes a certain function, namely $x A x$, since we wish to derive from this fact some properties of $x$ we examine how $x A x$ will change in the direction of its gradient, namely $2x A$. As defined in (9), $y$ is a translation from $x$ precisely in the direction of that gradient, the particular value of $\rho$ chosen is designed to keep $y$ within the "feasibility" set, i.e., $y A^2 y = 1$. We check next the last mentioned condition:

$$y A^2 y^T = (x - \rho x A) A^2 (x - \rho x A)^T =$$
$$= x A^2 x^T - 2 \rho x A^3 x^T + \rho^2 x A^4 x^T =$$
$$= 1 - 2 \rho \left[ x A^3 x^T - \frac{\rho}{2} (x A^4 x^T) \right]$$
$$= 1 - 2 \rho \left[ x A^3 x^T - (x A^3 x^T)(x A^4 x^T)^{-1}(x A^4 x^T) \right]$$
$$= 1 .$$

One can, incidentally, readily check that the particular value of $\rho$, as given in (8), is the only value of $\rho$ (other than $\rho = 0$) which yields $y A^2 y = 1$. Now,
since $y A^2 y^T = 1$, we must have, by definition of $\mu$,

(10) $y A y^T - x A x^T \geq 0$.

However,

$$y A y^T - x A x^T = (x - \rho x^A A)(x - \rho x^A)^T - x A x^T =$$

$$= -2 \rho x A^2 x^T + \rho^2 x A^3 x^T =$$

$$= 2 \rho \left[ \frac{\rho}{2} (x A^3 x^T) - (x A^2 x^T) \right] =$$

$$= 2 \rho (x A^4 x^T)^{-1} \left[ (x A^3 x^T)^2 - (x A^2 x^T)(x A^4 x^T) \right].$$

Thus, since $\rho > 0$, $(x A^4 x^T)^{-1} > 0$ and because (10) holds, we have:

(11) $(x A^3 x^T)^2 \geq (x A^2 x^T)(x A^4 x^T)$.

We now refer to inequality (3), which is a special case of (ii) - Theorem 1 with $A$ being the identity, letting $u = x A$, $v = x A^2$ we get:

(12) $(x A^3 x^T)^2 \leq (x A^2 x^T)(x A^4 x^T)$.

Combining (11) and (12), we get:

(13) $(x A^3 x^T)^2 = (x A^2 x^T)(x A^4 x^T)$.

However, from Lemma 1, again with $A$ being the identity matrix, we then know that $x A$, $x A^2$ are linearly dependent. Since $x A \neq 0$, it follows that there is a real number $\lambda$ such that $x A^2 = \lambda x A$, multiplying by $x^T : 1 = x A^2 x^T = \lambda x A x^T$, and $\lambda = (x A x^T)^{-1} = \mu^{-1}$. q.e.d.

As a final general result, we specialize (ii) - Theorem 1, and Lemma 1, for the case where $A$ is non-singular.
THEOREM 4

Let $A$ be p.s.d. and non-singular then,

$$\text{(14)} \quad (uv^T)^2 \leq (uAu^T)(vA^{-1}v^T) \quad \text{all } u, v \in \mathbb{R}^n$$

and equality holds above if and only if $u, vA^{-1}$ are dependent.

**PROOF:** We first note that $A^{-1}$ must be symmetric because $AA^{-1} = I$, thus $I^T = I = (AA^{-1})^T = (A^{-1})^T = (A^{-1})^T A$. But the inverse is unique, thus $A^{-1} = (A^{-1})^T$. Next, let $u, v \in \mathbb{R}^n$, we let

$$\text{(15)} \quad x = u, \quad y = vA^{-1}.$$

One readily checks that:

$$xAy^T = uv^T, \quad xAx^T = uAu^T, \quad yAy^T = vA^{-1}v^T.$$ 

Thus the desired inequality (14) follows from (ii) - Theorem 1. Now if (14) is actually an equation, then from Lemma 1, using $x, y$ as defined in (15), we get $u, vA^{-1}$ are linearly dependent. The converse also follows readily.

q.e.d.

**Note:** The condition of equality in (14) is directly connected with characteristic vectors of $A$ (and of course, those of $A^{-1}$), for suppose (14) is an equation and $u = v \neq 0$, then one sees immediately that $uA = \lambda u$ for some real number $\lambda$. The corresponding converse also holds in this case.

An iterative scheme, for deciding the definiteness of $A$, based on the proof of Theorem 3 might go as follows:

(a) By examining the diagonal elements of $A$ we have decided that, if at all, $A$ is p.s.d.

-10-
(b) We have an \( x \) such that \( xA \neq 0 \); if \( xA^2x \leq 0 \) then \( A \) is not p. s. d.
if \( xA^2x > 0 \) normalize \( x \) so that \( xA^2x = 1 \) and proceed to (c).

(c) We have an \( x \) such that \( xA \neq 0 \), \( xA^2x = 1 \); perform the transformation
given by (8) and (9). There are three cases:

Case 1. if \( yAy^T > xAx^T \) then \( A \) is not p. s. d.

Case 2. if \( yAy^T < xAx^T \) return to beginning of (c), using \( y \) as the
new "test" vector.

Case 3. if \( yAy^T = xAx^T \) we have isolated a characteristic vector
of \( A \), return to (b) using, as \( x \), a vector independent of all
characteristic vectors thus far obtained.

The preceding is, of course, "informal" in the sense that the iterative
procedure described above has not been shown to converge.
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