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SUPPORTS OF A CONVEX FUNCTION

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Let $C$ be a real, symmetric, $m \times m$, positive-semi-definite matrix. Let $R^m = \{(x_1, \ldots, x_m) \mid x_i \text{ is a real number, } i = 1, \ldots, m\}$, and let $K \subset R^m$ be a polyhedral convex cone, i.e., there exists a real $m \times n$ matrix $A$ such that $K = \{x \in R^m \text{ and } xA \leq 0\}$. Consider the function $\psi: K \to R$ defined by $\psi(x) = (x^TCx)^{1/2}$ for all $x \in K$. We wish to characterize the set, $U$, of all supports of $\psi$, where

\begin{equation}
U = R^m \cap \left\{ u \mid x \in K \implies u^Tx \leq (x^TCx)^{1/2} \right\}.
\end{equation}

Let $R^+_n = R^n \cap \{\pi \mid \pi \geq 0\}$ and consider the set

\begin{equation}
V = \left\{ v \mid \exists x \in R^m, \pi \in R^+_n \right. \\
\left. \quad \text{and } v = \pi A^T + xC, \ xC^T \leq 1, \ xA \leq 0 \right\}.
\end{equation}

We shall demonstrate:

**THEOREM:**

$U = V$.

We first show:

**LEMMA 1**

\[x, y \in R^m \implies (x^TCy)^2 \leq (x^TCx)(y^TCy)\,.

**Proof:** If $x, y \in R^m$ consider the polynomial $p(\lambda) = \lambda^2 x^TCx + 2\lambda x^TCy + y^TCy = (x + \lambda y)^TC(x + \lambda y)^T$. Since $C$ is positive-semi-definite, $p(\lambda) \geq 0$ for all real numbers $\lambda$, and thus the discriminant of $p$ is non-positive, i.e.,
\[ 4(xCy^T)^2 - 4(xCx^T)(yCy^T) \leq 0. \]

q.e.d.

As an immediate application of Lemma 1 we show:

**LEMMA 2**

\[ V \subseteq U \]

**Proof:** Let \( v \in V \), then there exist \( x \in \mathbb{R}^m, \pi \in \mathbb{R}^n_+ \) such that \( v = \pi A^T + xC \), \( xCx^T \leq 1 \). Now if \( y \in \mathbb{R}^m, yA \leq 0 \), then \( vy^T = yAx^T + xCy^T \) and \( vy^T \leq xCy^T \), because \( yA \leq 0, \pi^T \geq 0 \) and \( yAx^T \leq 0 \). Thus, \( vy^T \leq (xCx^T)^{\frac{1}{2}} (yCy^T)^{\frac{1}{2}} \), by Lemma 1, and \( vy^T \leq (yCy^T)^{\frac{1}{2}} \), because \( xCx^T \leq 1 \). Thus, \( v \in U \).

q.e.d.

From the fact that \( C \) is positive-semi-definite, it follows that:

**LEMMA 3**

The set \( V \) is convex.

**Proof:** If \( x_k \in \mathbb{R}^m, \pi_k \in \mathbb{R}^n_+ \), \( x_kA \leq 0 \), \( u_k = \pi_k A^T + x_kC, x_kCx_k^T \leq 1, \lambda_k \in \mathbb{R}^+ \) for \( k = 1, 2 \) and \( \lambda_1 + \lambda_2 = 1 \), then:

\[
\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 \pi_1 + \lambda_2 \pi_2)A^T + (\lambda_1 x_1 + \lambda_2 x_2)C, (\lambda_1 x_1 + \lambda_2 x_2)A \leq 0, \lambda_1 x_1 + \lambda_2 x_2 \in \mathbb{R}^m, \lambda_1 \pi_1 + \lambda_2 \pi_2 \in \mathbb{R}^n_+ ,
\]

and

\[
(\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - 1 \leq (\lambda_1 x_1 + \lambda_2 x_2)C(\lambda_1 x_1 + \lambda_2 x_2)^T - \lambda_1 x_1 Cx_1^T - \lambda_2 x_2 Cx_2^T - 
\]

\[
= -\lambda_1 \lambda_2 \left[ x_1 Cx_1^T - 2x_1 Cx_2^T + x_2 Cx_2^T \right] = 
\]

\[
= -\lambda_1 \lambda_2 (x_1 - x_2)C(x_1 - x_2)^T \leq 0, \text{ because } C \text{ is positive-semi-definite.}
\]
**Lemma 4**

The set $V$ is closed.

**Proof:** Let $\{w_k\}$ be a sequence with $w_k \in \mathbb{R}^m$, $k = 1, 2, \ldots$. We define the (pseudo) norm of $w_k$, denoted $\|w_k\|$, to be the smallest non-negative integer $p$ such that there exists a $k_0$ and for all $k \geq k_0$, $x_k$ has at most $p$ nonzero components. Now, suppose $u$ is in the closure of $V$, i.e., there exist sequences $\{u_k\}$, $\{v_k\}$ and $\{x_k\}$ such that

$$\begin{align*}
\pi_k \in \mathbb{R}^n, \quad x_k \in \mathbb{R}^m, \quad u_k = \pi_k A^T + x_k C \\
x_k A \leq 0 \quad \text{and} \quad y_k C x_k^T \leq 1,
\end{align*}$$

$k = 1, 2, \ldots$

and $\{u_k\}$ converges to $u$.

Suppose the sequence $\{x_k\}$ is bounded, then we may assume, having taken an appropriate subsequence, that for some $x \in \mathbb{R}^m$, $\{x_k\} \rightarrow x$ and thus, by (3), $xA \leq 0$ and $xCx^T \leq 1$. Now, $yA \leq 0 \Rightarrow u_k y^T - x_k C y^T = \pi_k A^T y^T = yA \pi_k \leq 0$, all $k \Rightarrow uy^T - xCy^T \leq 0$. Thus the system,

$$\begin{align*}
y \in \mathbb{R}^m \\
yA \leq 0 \\
(u - xC)y^T > 0
\end{align*}$$

has no solution and by the usual feasibility theorem for linear inequalities (see e.g. (4) or (5)) the system:
\[ \pi \in \mathbb{R}_+^n \]
\[ \pi A^T = u - xC \]

has a solution, and thus \( u \in V \).

We have just demonstrated that if \( \{ x_k \} \) is bounded, then \( u \in V \).

Since \( |\{ x_k \}| + |\{ x_k A \}| \leq m+n \), it is always possible to choose \( \{ x_k \} \) and \( \{ \pi_k \} \) satisfying (3) and such that \( |\{ x_k \}| + |\{ x_k A \}| \) is minimal.

We shall show next that if \( \{ x_k \}, \{ \pi_k \} \) are so chosen, then \( \{ x_k \} \) is indeed bounded, thus completing the proof. Suppose then that \( \{ x_k \} \) is not bounded, i.e., \( |x_k| = (x_k x_k^T)^{1/2} \to \infty \), and we may assume that \( |x_k| > 0 \) for all \( k \). Let

\[ z_k = \frac{x_k}{|x_k|}, \quad k = 1, 2, \ldots \]

then \( \{ z_k \} \) is bounded and we may assume that there is a \( z \in \mathbb{R}^m \) such that the \( z_k \) converge to \( z \) and \( |z| = 1 \). From (3) it follows that \( z_k A \leq 0 \) and \( z_k C z_k^T \leq \frac{1}{|x_k|^2} \) for all \( k \). Thus, \( zA \leq 0 \) and \( zCz^T \leq 0 \). But then, from Lemma 1, \( zCy^T = 0 \) for all \( y \in \mathbb{R}^m \), and \( zC = 0 \). Summarizing:

\[ z \in \mathbb{R}^m, zA \leq 0, zC = 0. \]

Note that if \( z \) has a nonzero component, then infinitely many \( x_k \)'s must have the same component nonzero, this follows from the fact that \( z \) is the limit of \( \frac{x_k}{|x_k|} \). As a consequence, if \( \{ \lambda_k \} \) is any sequence of real numbers, then \( \left| \left\{ x_k + \lambda_k z \right\} \right| \leq \left| \left\{ x_k \right\} \right| \). If \( zA \neq 0 \), and \( a^j, j = 1, \ldots, n, \)
denotes the \( j \)-th column of \( A \), let
\[
\lambda_k = \max_j - \frac{x_k a^j}{z a^j} \quad \text{subject to} \quad z a^j < 0
\]
Then we may replace, in (3), \( x_k \) by \( x_k + \lambda_k z \) because \( \lambda_k z a^j + x_k a^j \leq 0 \) for all \( j \), and \( (x_k + \lambda_k z)A \leq 0 \). Also \( zC = 0 \) and thus \( (x_k + \lambda_k z)C = x_k C \),
\[
(x_k + \lambda_k z)C(x_k + \lambda_k z)^T = x_k C x_k^T \leq 1.
\]
However each \( (x_k + \lambda_k z)A \) has at least one more zero component than \( x_k A \), contradicting the minimality of \( I \).

Lastly, we show:

**Lemma 5**

\[ U \subset V \]

**Proof:** Suppose \( u \notin V \). By Lemma 3 and 4, \( V \) is a closed convex set, hence there is a hyperplane which separates \( u \) strongly from \( V \) (see [4]). Thus there exist \( x \in \mathbb{R}^m \) and \( a \in \mathbb{R} \) such that
\[
ux^T > a \geq vx^T \quad \text{all} \quad v \in V.
\]
Now, if \( \pi \in \mathbb{R}_+^n \) then \( \mathbf{v} = \pi A^T \) is in \( \mathbf{V} \) (taking \( x = 0 \) in the definition of \( \mathbf{V} \)).

Thus \( xA^T = \pi A^T x \leq a \) for all \( \pi \in \mathbb{R}_+^n \), and \( xA \leq 0, \ x \in K \). Also \( \mathbf{v} = 0 \) is in \( \mathbf{V} \), so that \( a \geq 0 \). If \( u \in \mathbf{U} \) then \( 0 \leq a < ux^T \leq (x^T x)^{1/2} \), thus \( x^T x > 0 \) and

\[
\mathbf{v} = \frac{x^T}{(x^T x)^{1/2}} \in \mathbf{V},
\]

consequently,

\[
(x^T x)^{1/2} > a \geq \frac{x^T}{(x^T x)^{1/2}} = (x^T x)^{1/2}
\]

a contradiction. Thus \( u \notin \mathbf{U} \). q.e.d.

Note: A direct application of Lemmas 2 and 5 yields the theorem stated at the beginning.
REFERENCES


