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THE POTENTIAL PROBLEM OF THE OPTIMUM
PROPELLER WITH FINITE HUB

by

A.J. Tachmindji

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August 1956
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\[ \mu_0 = \frac{1}{\lambda_i} \frac{\omega R}{v} \]

\[ \nu = \frac{p}{2} (2m + 1) \]

\[ \phi \quad \text{Velocity potential} \]

\[ \phi_1 \quad \text{Velocity potential defined by Equation [6]} \]

\[ \omega \quad \text{Propeller angular velocity} \]

Subscripts

\[ 0 \quad \text{Pertaining to quantities at } r = R \]

\[ 1 \quad \text{Pertaining to quantities at the hub} \]

Primes denote differentiation with respect to the argument.
ABSTRACT

The potential problem is solved for the circulation distribution of an optimum propeller with a finite number of blades and a hub of constant diameter. The effect of the hub has been calculated for specific cases, showing that it becomes important for propellers with large hub diameters and small number of blades and increases with increasing pitch.

INTRODUCTION

The determination of the bound circulation for an optimum propeller was initially performed by Goldstein who was able to relate the circulation distribution of a propeller with a finite number of blades to one having an infinite number. Goldstein's solution was, however, performed for the case of a propeller having a zero hub diameter and when the vortex sheet extends to the propeller axis. Such a solution is considered sufficiently accurate for propellers with a hub diameter which is relatively small compared with the propeller diameter; in this case, the presence of the hub is assumed to have little effect on the circulation distribution along the radius.

However, the increasing use of propellers with relatively large hubs has emphasized the need of a solution for such cases. The problem was originally investigated by Lerbs who, through the use of corrective "induction factors," was able to consider the presence of an infinitely long hub. A solution to the potential problem has also been given by McCormick for the effect of a finite hub but with simplified boundary conditions given at the hub radius. The effect of such assumptions is to produce a discontinuous change of circulation at the hub which is contrary to available experimental evidence.

The problem is solved for a propeller having a minimum energy loss and for which the flow far behind the propeller can be considered to be the same as that formed by rigid trailing vortices moving backwards at constant angular velocity. The hub is assumed to be of constant diameter and extending from the propeller plane to infinity.

STATEMENT OF THE PROBLEM

The problem is analogous to that of a propeller with zero hub but with modified boundary conditions. In this case, however, solutions which have singularities at the shaft axis can also be considered.

---

References are listed on page 16.
Laplace's equation for this type of flow can be reduced to

\[
\left( \mu \frac{\partial}{\partial \mu} \right)^2 \phi + \left( 1 + \mu^2 \right) \frac{\partial^2 \phi}{\partial \zeta^2} = 0
\]  

where \( \phi \) is the velocity potential,  
\( \mu \) is equal to \( \frac{\omega r}{v} \),  
\( \omega \) is the angular velocity,  
\( v \) is the velocity of advance,  
\( r \) is the distance from axis of rotation,  
\( \zeta \) is equal to \( \theta - \frac{\omega z}{v} \), and  
\( r, \theta, z \) are cylindrical coordinates.

The boundary condition which has to be satisfied on the vortex sheets is that the velocity be normal to the sheet and can be written as

\[
\frac{\partial \phi}{\partial \zeta} = -\frac{\mu^2}{1+\mu^2} \quad \text{on} \quad \zeta = 0 \quad \text{or} \quad \frac{2\pi}{p}
\]  

for \( r_1 < r < R \) or \( \mu_1 < \mu < \mu_0 \)

where \( p \) is the number of blades,  
\( R \) is the propeller radius,  
\( r_1 \) is the hub radius,  
\( \mu_0 \) is equal to \( \frac{\omega R}{v} \), and  
\( \mu_1 \) is equal to \( \frac{\omega r_1}{v} \)

Additional conditions have to be imposed at \( r = R \) where continuity should exist for both \( \phi \) and \( \text{grad } \phi \). Furthermore, \( \text{grad } \phi \) should also vanish at an infinitely large radius.

The presence of the hub imposes certain boundary conditions on both the circulation at the vortex sheet and on the induced flow. When approaching \( r_1 \) from inside the hub, the circulation has to be zero. When, however, \( r_1 \) is approached from outside the hub, the possibility exists that the circulation may change either discontinuously or continuously to zero. A discontinuity in the circulation distribution at the hub will, however, mean that a discontinuity will exist in the pressure field between two adjacent blades. Physical considerations preclude the possibility of the existence of such pressure changes, and the flow will tend to
equalize from the pressure side of one blade to the suction side of the other. Furthermore, available experimental information\(^4\) indicates that the radial distribution of circulation is continuous to zero at the hub. Hence, the boundary condition is written as

\[
\phi = 0 \text{ at } r = r_1 \text{ or } \mu = \mu_1
\]  

[3]

The presence of the hub also requires that the radial component of the velocity be zero. By this condition

\[
\frac{\partial \phi}{\partial \mu} = 0 \text{ at } r = r_1 \text{ or } \mu = \mu_1
\]  

[4]

\section*{SOLUTION OF THE POTENTIAL PROBLEM}

\section*{THE EXTERNAL FIELD, \(r > R\)}

The conditions of the problem are such that \(\phi\) has to be an odd function of \(\zeta\), single-valued and continuous. Assuming that the solution can be expressed in a sine series, we find that the coefficient of \(\sin p_n \zeta\) must be a linear function of \(I_{p_n}(\mu_1 p_n)\) and \(K_{p_n}(\mu_0 p_n)\), where \(I\) and \(K\) are the modified Bessel functions. But \(I_{p_n}(\mu_1 p_n)\) cannot occur since \(\text{grad } \phi\) must vanish when \(r, \text{ or } \mu, \text{ is infinite. A solution independent of } \zeta\) can also exist which satisfies Equation [1] and which is in the form of \(\log \mu\). Hence, we may assume

\[
\phi = \sum_{n=1}^{\infty} \frac{2 c_n}{p} \frac{K_{p_n}(\mu_1 p_n)}{K_{p_n}(\mu_0 p_n)} \sin p_n \zeta + c \log \frac{\mu}{\mu_0}
\]  

[5]

where \(c_n\) are constants to be determined with \(K_{p_n}(\mu_0 p_n)\) and \(\log \mu_0\) being inserted in the denominator for future convenience.

\section*{THE INTERNAL FIELD, \(r_1 < r < R\)}

In this region we write

\[
\phi = -\frac{\mu^2}{1+\mu^2} \zeta + \phi_1
\]  

[6]

where \(\phi_1\) is a velocity potential to be determined. For such a case the boundary condition given by Equation [2], becomes

\[
\frac{\partial \phi_1}{\partial \zeta} = 0 \text{ at } \zeta = 0 \text{ and } \frac{2\pi}{p}
\]

and Laplace's equation
\[
\left(\mu \frac{\partial}{\partial \mu} \right)^2 \varphi_1 + \left(1 + \mu^2\right) \frac{\partial^2 \varphi_1}{\partial \zeta^2} = \left(\mu \frac{\partial}{\partial \mu} \right)^2 \left(\frac{\mu^2}{1 + \mu^2}\right) \zeta
\]

Since \( \zeta \) is periodic in the interval 0 to \( 2\pi/p \), it can be expanded in a half-range cosine series

\[
\zeta = \frac{\pi}{p} - \frac{8}{\pi p} \sum_{m=0}^{\infty} \frac{\cos((2m + 1) \frac{p\zeta}{2})}{(2m + 1)^2}
\]

and for the same interval, \( \varphi_1 \) can be expanded in a semi-infinite cosine series in \( \zeta \)

\[
\varphi_1 = f_0(\mu) + \sum_{m=0}^{\infty} f_m(\mu) \cos((2m + 1) \frac{p\zeta}{2})
\]

Differentiating term by term and substituting in Equation [7]

\[
\left(\mu \frac{\partial}{\partial \mu} \right)^2 f_0(\mu) = \frac{\pi}{p} \left(\mu \frac{\partial}{\partial \mu} \right)^2 \left(\frac{\mu^2}{1 + \mu^2}\right)
\]

and

\[
\left(\mu \frac{\partial}{\partial \mu} \right)^2 f_m(\mu) - (1 + \mu^2) \frac{(2m + 1)^2}{4} f_m(\mu) = -\frac{8}{\pi p} \frac{1}{(2m + 1)^2} \left(\mu \frac{\partial}{\partial \mu} \right)^2 \left(\frac{\mu^2}{1 + \mu^2}\right)
\]

Equation [10] can be integrated to give

\[
f_0(\mu) = \frac{\pi}{p} \left(\frac{\mu^2}{1 + \mu^2}\right) + D \log \frac{\mu}{\mu_0}
\]

where \( D \log \mu_0 \) is an arbitrary constant. We now consider a new function \( g_m(\mu) \), such that

\[
f_m(\mu) = -\frac{8}{\pi p} \frac{1}{(2m + 1)^2} \left[\frac{\mu^2}{1 + \mu^2} - g_m(\mu)\right]
\]


\[
\left(\mu \frac{\partial}{\partial \mu} \right)^2 g_m(\mu) - (2m + 1)^2 (1 + \mu^2) \frac{p^2}{4} g_m(\mu) = (2m + 1)^2 \frac{p^2}{4} \mu^2
\]

The solution of this homogeneous differential equation can be recognized as a Bessel function, while the particular solution can be written as a Lommel function. The general solution for \( g_m(\mu) \) is
where
\[ \nu = \frac{p}{2} (2m + 1) \]

\[ T_{1,\nu}(\nu \mu) = \frac{\mu^2}{1 + \mu^2} + \frac{16 \mu^2 (1 - \mu^2)}{p^2 (2m + 1)^2 (1 + \mu^2)^4} \]
\[ + \frac{256 \mu^2 (1 - 14 \mu^2 + 21 \mu^4 - 4 \mu^6)}{p^4 (2m + 1)^4 (1 + \mu^2)^7} + \ldots \ldots \]

\[ a_m, b_m = \text{arbitrary constants} \]

Substituting the values for \( g_m(\mu), f_0(\mu), f_m(\mu) \) into Equation [9] and hence into Equation [6] we get
\[ \phi = \frac{2}{p} \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu)}{(2m + 1)^2} + a_m \frac{I_{\nu}(\nu \mu)}{I_{\nu}(\nu \mu_0)} + b_m \frac{K_{\nu}(\nu \mu)}{K_{\nu}(\nu \mu_0)} \right] \cos \nu \zeta + D \log \frac{\mu}{\mu_0} \]

where \( a_m \) and \( b_m \) are arbitrary constants to be determined from the boundary conditions.

**DETERMINATION OF THE ARBITRARY CONSTANTS**

The arbitrary constants \( a_m, b_m, c_n, c, \) and \( D \) are determined from the following boundary conditions:

(a) \( \phi \) is continuous at \( r = R \)

(b) \( \frac{\partial \phi}{\partial \mu} \) is continuous at \( r = R \)

(c) \( \phi = 0 \) at \( r = r_1 \)

(d) \( \frac{\partial \phi}{\partial \mu} = 0 \) at \( r = r_1 \)

In order to evaluate the constants, it is first necessary to express \( \phi \) for the external field in terms of \( \sin p_n \zeta \) in the interval \( 0 < \zeta < 2\pi/p \). In this range we write
\[ \cos \nu \zeta = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - (2m + 1)^2} \sin p_n \zeta \]
Substituting Equation [18] in Equation [17] and equating to Equation [5] for \( \mu = \mu_0 \) and all \( \zeta \), we satisfy boundary condition (a) when

\[
\frac{8}{\pi} \sum_{m=0}^{\infty} \frac{n}{4n^2 -(2m+1)^2} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu_0)}{(2m+1)^2} + a_m + b_m \right] = c_n \quad [19]
\]

Similarly for boundary condition (b),

\[
\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(2m+1)}{4n^2 -(2m+1)^2} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu_0)}{(2m+1)^2} + a_m \frac{I_{\nu}(\nu \mu_0)}{I_{\nu}(\nu \mu_0)} + b_m \frac{K_{\nu}(\nu \mu_0)}{K_{\nu}(\nu \mu_0)} \right] = c_n \frac{K_{\rho n}(p \mu \mu_0)}{K_{\rho n}(p \mu \mu_0)} \quad [20]
\]

and

\[ D = c \]

where primes denote differentiation with respect to the argument.

For zero circulation at the hub

\[
\sum_{m=0}^{\infty} \frac{n}{4n^2 -(2m+1)^2} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu_1)}{(2m+1)^2} + a_m \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} + b_m \frac{K_{\nu}(\nu \mu_1)}{K_{\nu}(\nu \mu_0)} \right] = 0 \quad [21]
\]

and from (d)

\[
\sum_{m=0}^{\infty} \frac{n}{4n^2 -(2m+1)^2} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu_1)}{(2m+1)^2} + a_m \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} + b_m \frac{K_{\nu}(\nu \mu_1)}{K_{\nu}(\nu \mu_0)} \right] + \frac{D}{\mu_1} = 0 \quad [22]
\]

Equations [19], [20], [21], and [22] can be solved for the coefficients \( a_m, b_m, c_n, \) and \( D \). From Equation [19] and [20], we obtain

\[
\sum_{m=0}^{\infty} \frac{a_m}{4n^2 -(2m+1)^2} \left[ \frac{(2m+1)}{2n} \frac{I_{\nu}(\nu \mu_0)}{I_{\nu}(\nu \mu_0)} - 2n \frac{K_{\rho n}(p \mu \mu_0)}{K_{\rho n}(p \mu \mu_0)} \right] + \sum_{m=0}^{\infty} \frac{b_m}{4n^2 -(2m+1)^2} \left[ \frac{(2m+1)}{2n} \frac{K_{\nu}(\nu \mu_0)}{K_{\nu}(\nu \mu_0)} - 2n \frac{K_{\rho n}(p \mu \mu_0)}{K_{\rho n}(p \mu \mu_0)} \right] = 0 \quad [23]
\]
The determination of the constants $a_m$ and $b_m$ is now considered from Equations [21] and [23]. A method of successive approximations can be used to determine the coefficients from this infinite set of simultaneous equations. However, the singularity which exists at the edges $\zeta = 0$ and $2 \pi / \rho$ will result in a very slow convergence. Restricting the solution for values of $\mu_0$ which are not too small, we can write from Equation [16]

$$T_{1, \nu}(\nu \mu_0) = \frac{\theta}{1 + \mu_0^2}$$

This approximation improves with increasing number of blades. Also

$$\nu \cdot T_{1, \nu}(\nu \mu_0) \approx \frac{2 \mu_0}{(1 + \mu_0^2)^2}$$

Furthermore, using Nicholson's asymptotic expansions

$$\frac{K_{p_n}^*(p_n \mu_0)}{K_{p_n}(p_n \mu_0)} = \frac{(1 + \mu_0^2)^{3/2}}{2 \nu (1 + \mu_0^2)} + \frac{\mu_0 (4 - \mu_0^2)}{8 \nu (1 + \mu_0^2)^{5/2}} - \ldots$$

or for $\mu_0$ not too small

$$\frac{K_{p_n}^*(p_n \mu_0)}{K_{p_n}(p_n \mu_0)} \approx \frac{(1 + \mu_0^2)^{3/2}}{\mu_0}$$

Similarly

$$\frac{I_{\nu}^*(\nu \mu_0)}{I_{\nu}(\nu \mu_0)} = \frac{(1 + \mu_0^2)^{3/2}}{2 \nu (1 + \mu_0^2)} + \frac{\mu_0 (4 - \mu_0^2)}{8 \nu (1 + \mu_0^2)^{5/2}} - \ldots$$

$$\approx \frac{(1 + \mu_0^2)^{3/2}}{\mu_0}$$

Equation [23] can then be written

$$\sum_{m = 0}^{\infty} \left[ \frac{a_m}{2n - (2m + 1)} + \frac{b_m}{2n + (2m + 1)} \right] = \frac{4 \mu_0}{\pi} \sum_{m = 0}^{\infty} \frac{1}{[4n^2 - (2m + 1)^2] (2m + 1)^2} \left[ \frac{2n + \frac{4}{p (1 + \mu_0^2)^{3/2}}} {p (1 + \mu_0^2)^{3/2}} \right]$$
For usual values of $\mu_0$, it is possible to neglect the second term on the right-hand side of (23) as compared with the first. Then,

$$
\sum_{m=0}^{\infty} \left[ \frac{a_m}{2n-(2m+1)} + \frac{b_m}{2n+(2m+1)} \right] = -\mu_0^2 \frac{\pi}{1+\mu_0^2} \frac{\pi}{4n}
$$

and from Equation (21)

$$
b_m = -\frac{K_v(\nu \mu_0)}{K_v(\nu \mu_1)} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu_1)}{(2m+1)^2} + a_m I_{\nu}(\nu \mu_0) \right]
$$

and finally we obtain

$$
\sum_{m=0}^{\infty} \frac{a_m}{2n-(2m+1)} \left[ 1 - \frac{2n-(2m+1) K_v(\nu \mu_0)}{2n+(2m+1) K_v(\nu \mu_1)} \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} \right] = -\mu_0^2 \frac{\pi}{1+\mu_0^2} \frac{\pi}{4n} + \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{T_{1,\nu}(\nu \mu_1)}{2n+(2m+1)^2} \frac{K_v(\nu \mu_0)}{K_v(\nu \mu_1)}
$$

for $n=1, 2, 3, \ldots$

The existence of a solution for this infinite system cannot be shown, as the necessary theorems have conditions which are too restrictive in this case. However, the system is convergent, and a solution can be found by the standard method of successive approximations.

It will be noted that as the hub diameter tends to zero, i.e., $\mu_1$ tends to zero, Equation [27] degenerates into a system of equations given by Goldstein.

**DETERMINATION OF THE Bound CIRCULATION**

The circulation distribution is proportional to the discontinuity in velocity potential at the screw surface $\zeta=0$ or $2\pi/\rho$. If $[\phi]$ denotes the discontinuity in the potential, then from Equation [17]

$$
[\phi] = \frac{4}{\rho} \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu)}{(2m+1)^2} + a_m I_{\nu}(\nu \mu_0) + b_m \frac{K_v(\nu \mu_1)}{K_v(\nu \mu_0)} \right]
$$

It is then possible to write

$$
\frac{p \omega \Gamma}{2\pi \nu \nu} = \frac{p}{2\pi} [\phi]
$$

$$
= \frac{2}{\pi} \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu \mu)}{(2m+1)^2} + a_m I_{\nu}(\nu \mu_0) + b_m \frac{K_v(\nu \mu_1)}{K_v(\nu \mu_0)} \right]
$$

[28]
If $\Gamma_\infty$ is the circulation of a blade section for a propeller having an infinite number of blades, then

$$\frac{p \omega \Gamma_\infty}{2\pi \omega} = \frac{\mu^2}{1 + \mu^2}$$

Defining $\kappa_h = \frac{\Gamma_h}{\Gamma_\infty}$, where $\kappa_h$ is the factor for a propeller having a finite hub,

$$\kappa_h = \frac{2}{\pi} \left(1 + \mu^2\right) \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} + a_m \frac{I_{\nu}(\nu\mu)}{I_{\nu}(\nu\mu_0)} + b_m \frac{K_{\nu}(\nu\mu)}{K_{\nu}(\nu\mu_0)} \right]$$

or from Equation [26]

$$\kappa_h = \frac{8}{\pi^2} \left(1 + \mu^2\right) \sum_{m=0}^{\infty} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} \left[ \frac{1 - \frac{T_{1,\nu}(\nu\mu_1)}{T_{1,\nu}(\nu\mu)}}{\frac{K_{\nu}(\nu\mu)}{K_{\nu}(\nu\mu_1)}} \right]$$

$$+ \frac{2}{\pi} \left(1 + \mu^2\right) \sum_{m=0}^{\infty} a_m \frac{I_{\nu}(\nu\mu)}{I_{\nu}(\nu\mu_0)} \left[ \frac{1 - \frac{K_{\nu}(\nu\mu)}{K_{\nu}(\nu\mu_1)}}{\frac{I_{\nu}(\nu\mu)}{I_{\nu}(\nu\mu_1)}} \right]$$

[29]

**EFFECT OF FINITE HUB**

The use of Equation [27] for the determination of the coefficients $a_m$ involves considerable labor, and in order to indicate the effect of the hub, an alternative set of equations has been derived in the Appendix. These equations involve simplifications of the basic development but have good accuracy for the usual values of $\mu_0$ and for hub diameters not larger than half the propeller diameter. The errors which are introduced decrease considerably with increasing number of blades.

Through the use of Equations [31], [39], [40], and [43] of the Appendix, calculations were performed showing the effect of hub diameters of 0.2, 0.3 and 0.4 for a four-bladed propeller. Both the factor $\kappa$ and the circulation distribution are shown on Figures 1 and 2 for this propeller operating at $1/\lambda_i = \mu_0 = 4.0$. Note that the circulation function falls off to zero at the hub corresponding to the boundary condition (c) for the circulation. Near the tip, the distributions for finite and zero hub coincide; this is to be expected since the influence of the hub must decrease towards the tip. Figure 3 shows the ratio of the factor $\kappa$ for a finite hub and a zero hub for various hub diameters.

The effect of number of blades on $\kappa$ is shown on Figure 4 for a two-, a four- and a six-bladed propeller operating at $1/\lambda_i = \mu_0 = 4.0$ for $r/R = 0.2$. Note that the effect diminishes with increasing number of blades.

It should be noted that although the presence of the hub has an appreciable effect on the circulation near the hub, the velocities are the same over the entire radius. Both the condition of normality and Betz's condition hold for a finite hub, and they are sufficient to uniquely determine the induced velocity components.
Figure 1 - Effect of Hub Diameter for $\mu_0 = 4$, $p = 4$

Figure 2 - Effect of Hub Diameter on Distribution of Circulation for $\mu_0 = 4$, $p = 4$
Figure 3 - Effect of Hub Diameter for $\kappa_0/\kappa_h$ for $\mu_0 = 4$, $p = 4$

Figure 4 - Effect of Number of Blades for $\mu_0 = 4$, $r_1/R = 0.2$
CONCLUSION

The solution of the potential problem for a propeller with finite hub indicates that the presence of the hub results in a decreased circulation at the inner radii. This effect becomes important for propellers with large hub diameters or small number of blades and increases with increasing pitch.

APENDIX

APPROXIMATE SOLUTION FOR THE COEFFICIENTS

The solution of Equation [27] for the coefficient $a_m$ involves considerable labor, and an attempt has been made to approximate the solution of this system for usual values of $\mu_0$. For this purpose, let

$$a_m = -\frac{\mu_0^2}{1 + \mu_0^2} A_m + \delta A_m$$  \[30\]

where $A_m$ is the coefficient defined by Goldstein\(^1\) and is given by

$$A_m = \frac{(2m)!}{2^m m! (2m+1)}$$  \[31\]

and $\delta A_m$ is a correction coefficient to be determined. Furthermore, Nicholson's asymptotic expansions\(^6\) are used for the $I$ and $K$ functions, where

$$I_\nu(\nu \mu) = \left(\frac{1}{2\pi \nu}\right)^{\frac{3}{2}} \frac{1}{(1 + \mu^2)^{\frac{1}{2}}} e^{\nu y}$$  \[32\]

and

$$K_\nu(\nu \mu) = \left(\frac{1}{2\pi \nu}\right)^{\frac{3}{2}} \frac{1}{(1 + \mu^2)^{\frac{1}{2}}} e^{-\nu y}$$  \[33\]

where

$$y = \sqrt{1 + \mu^2} - \frac{1}{2} \log \frac{\sqrt{1 + \mu^2} + 1}{\sqrt{1 + \mu^2} - 1}$$  \[34\]

Substitution of Equations [30], [32], and [38] into the system of Equation [27], gives
\[
\sum_{m=0}^{\infty} \delta A_m \left[ \frac{1}{-(2m+1)} - e^{2\nu(y_0-y_1)} \right] = \frac{4}{\pi} \left( \frac{1+\mu_1^2}{1+\mu_0^2} \right)^{\frac{3}{2}} \sum_{m=0}^{\infty} e^{-\nu(y_0-y_1)} T_{1,\nu}(\nu\mu_1)
\]
\[
\sum_{m=0}^{\infty} \frac{\delta A_m}{2n-(2m+1)} = \frac{4}{\pi} \left( \frac{1+\mu_1^2}{1+\mu_0^2} \right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{T_{1,\nu}(\nu\mu_1)}{2n+(2m+1)} \left( \frac{2}{2m+1} \right)^2
\]

For usual values of \( \mu_0 \) and for values of \( r_1/R \) which are smaller than 0.5, \( A_m e^{-\nu(y_0-y_1)} \) is small compared with \( T_{1,\nu}(\nu\mu_1)/(2m+1)^2 \). Furthermore, the second term of the left-hand side of Equation [35] is also small compared with the first and can safely be neglected. Equation [35] can then be written

\[
\sum_{m=0}^{\infty} \frac{\delta A_m}{2n-(2m+1)} = 4 \left( \frac{1+\mu_1^2}{1+\mu_0^2} \right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{T_{1,\nu}(\nu\mu_1)}{2n+(2m+1)} \left( \frac{2}{2m+1} \right)^2
\]

Using Goldstein's definition of the \( T \) function as

\[
T_{1,\nu}(\nu\mu_1) = \frac{\mu_1^2}{1+\mu_1^2} \frac{F_{p_1}}{2} (\mu_1)
\]

it is found that the terms corresponding to values of \( m = 1, 2, 3 \ldots \) of the right hand side of equation can be omitted with good accuracy. The error introduced by such an approximation is less than 3-percent in the determination of the value of \( A_m \) for a two-bladed propeller, \( r_1/R = 0.4 \) and \( \mu_0 = 3 \). The error decreases for increasing number of blades and increasing \( \mu_0 \).

Equation [36] becomes

\[
\sum_{m=0}^{\infty} \frac{\delta A_m}{2n-(2m+1)} = 4 \left( \frac{1+\mu_1^2}{1+\mu_0^2} \right)^{\frac{3}{2}} \left[ \frac{\mu_1^2}{1+\mu_1^2} \frac{F_{p_1}}{2} (\mu_1) \right] \frac{e^{-\frac{p}{2}(y_0-y_1)}}{(2n+1)}
\]

for \( n = 1, 2, 3 \ldots \)

The \( F \) functions for two and four blades are plotted on Figures 5 and 6. Let

\[
\delta A_m = \frac{4}{\pi} \left( \frac{1+\mu_1^2}{1+\mu_0^2} \right)^{\frac{3}{2}} \left[ \frac{\mu_1^2}{1+\mu_1^2} \frac{F_{p_1}}{2} (\mu_1) \right] e^{-\frac{p}{2}(y_0-y_1)} B_m
\]

then \( B_m \) has to satisfy the equation

\[
\sum_{m=0}^{\infty} \frac{B_m}{2n-(2m+1)} = \frac{1}{2n+1} \quad \text{for} \quad n = 1, 2, 3 \ldots
\]
Solution of the system [40] gives

\[ B_0 = 0.4762 \quad B_5 = 0.0136 \]
\[ B_1 = 0.1128 \quad B_6 = 0.0095 \]
\[ B_2 = 0.0531 \quad B_7 = 0.0066 \]
\[ B_3 = 0.0200 \quad B_8 = 0.0044 \]
\[ B_4 = 0.0200 \quad B_9 = 0.0025 \]

When \( a_m \) is represented by Equation [30], the equation for the factor \( \kappa_h \), becomes

\[
\kappa_h = \frac{2}{\pi} \left( \frac{1 + \mu^2}{\mu^2} \right) \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{\nu^2}{(2m+1)^2} - \frac{\mu_0^2}{1 + \mu_0^2} A_m \frac{I_{\nu}(\nu \mu)}{I_{\nu}(\nu \mu_0)} \right] - \frac{2}{\pi} \left( \frac{1 + \mu^2}{\mu^2} \right) \sum_{m=0}^{\infty} \left[ \frac{4}{\pi} \frac{\nu^2}{(2m+1)^2} - \frac{\mu_0^2}{1 + \mu_0^2} A_m \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} \right] \frac{K_{\nu}(\nu \mu)}{K_{\nu}(\nu \mu_1)}
\]

\[ [41] \]

\[
- \frac{2}{\pi} \left( \frac{1 + \mu^2}{\mu^2} \right) \sum_{m=0}^{\infty} \delta A_m \left[ \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} \frac{K_{\nu}(\nu \mu)}{K_{\nu}(\nu \mu_1)} - \frac{I_{\nu}(\nu \mu)}{I_{\nu}(\nu \mu_0)} \right]
\]

It is noticed that the first term of Equation [41] is equal to the ratio of the circulations for a propeller having zero hub. Denoting this by \( \kappa_0 \) and writing

\[
\kappa_h = \kappa_0 - \delta \kappa
\]

then

\[
\delta \kappa = \frac{2}{\pi} \left( \frac{1 + \mu^2}{\mu^2} \right) \sum_{m=0}^{\infty} \left[ \frac{K_{\nu}(\nu \mu)}{K_{\nu}(\nu \mu_1)} \left[ \frac{4}{\pi} \frac{\nu^2}{(2m+1)^2} - A_m \frac{\mu_0^2}{1 + \mu_0^2} \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} \right] \right] + \delta A_m \left[ \frac{I_{\nu}(\nu \mu_1)}{I_{\nu}(\nu \mu_0)} \frac{K_{\nu}(\nu \mu)}{K_{\nu}(\nu \mu_1)} - \frac{I_{\nu}(\nu \mu)}{I_{\nu}(\nu \mu_0)} \right]
\]

[43]

The factor \( \delta \kappa \) can, therefore, be considered as a correction factor to the case of a propeller with zero hub and can be evaluated using Equations [31], [39] and [40] for the coefficients \( A_m \) and \( \delta A_m \).
Figure 5 - Function $F$ for $p = 2$

Figure 6 - Function $F$ for $p = 4$
REFERENCES


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The potential problem is solved for the circulation distribution of an optimum propeller with a finite number of blades and a hub of constant diameter. The effect of the hub has been calculated for specific cases, showing that it becomes important for propellers with large hub diameters and small number of blades and increases with increasing pitch.

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