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 VOLUME II

 CONTRIBUTIONS TO PROBABILITY THEORY

 EDITED BY JERZY NEYMAN

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1. Introduction

Consider, for example, a classical mechanical system with Lagrangian

\[ L(x, \dot{x}) = \frac{\dot{x}^2}{2} - U(x). \]

The wave function of the quantum mechanical system corresponding to this classical one changes with time \( t \) according to the Schrödinger equation

\[ \frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial x^2} - U \psi, \quad \psi(0+, x) = \varphi(x). \]

Feynman [3] expressed this wave function \( \psi(t, x) \) in the following integral form, which we shall here call the Feynman integral

\[ \psi(t, x) = \frac{1}{N} \int_{\Gamma_x} \exp \left\{ \frac{i}{\hbar} \int_0^t \left[ \frac{\dot{x}_r^2}{2} - U(x_r) \right] \, dr \right\} \varphi(x_0) \prod_r dx_r, \]

where \( \Gamma_x \) is the space of paths \( X = (x, 0 < \tau \leq t) \) with \( x_0 = x \), \( \prod_r dx_r \) is a uniform measure on \( R^{(0,t)} \), and \( N \) is a normalization factor. It should be noted that the integral \( \int_0^t \left[ \dot{x}_r^2/2 - U(x_r) \right] \, dr \) is the classical action integral along the path \( X \). (This idea goes back to Dirac [1].) It is easy to see that (1.3) solves (1.2) unless we require mathematical rigor. It is our purpose to define the generalized measure \( \prod_r dx_r/N \), that is, the integral \( \int_{\Gamma_x} F(X) \prod_r dx_r/N \), rigorously and to prove that (1.3) solves (1.2) in case \( U(x) \equiv 0 \) (case of no force) or \( U(x) \equiv x \) (case of constant force). See theorem 5.2 and theorem 5.3 below. We hope this fact will be proved for a general \( U(x) \) with some appropriate regularity conditions.

Our definition is also applicable to the Wiener integral; namely, using it, we shall prove that the solution of the heat equation

\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u, \quad u(0+, x) = f(x), \]

is given by

\[ u(t, x) = \frac{1}{N} \int_{\Gamma_x} \exp \left\{ - \int_0^t \left[ \frac{\dot{x}_r^2}{2} + U(x_r) \right] \, dr \right\} f(x_0) \prod_r dx_r. \]
for any bounded continuous function $U(x)$. See theorem 4.3. This should be called the Feynman version of Kac's theorem that

$$(1.5') \quad u(t, x) = \int_{\mathbb{R}} \exp \left[ -\int_0^t U(x, \tau) \, d\tau \right] f(x_0) W_x(dX)$$

solves (1.4). In this paper the paths, that is, the points in $\Gamma_x$, are denoted with capital letters $X, Y, \cdots$ and their values at time $\tau$ are denoted with the corresponding small letters with the suffix $\tau$ such as $x_\tau, y_\tau, \cdots$. Now that Kac's theorem is well known to probabilists, no one bothers with its Feynman version. However, it is interesting that Kac had the Feynman version (1.5) in mind and formulated it as (1.5') to make it rigorous [5].

Gelfand and Yaglom [4] proposed a method of defining the Feynman integral. They replaced $A$ with $A - i\sigma$ in (1.3) to reduce the Feynman integral to the Wiener integral and defined the Feynman integral as a limit of the Wiener integral by letting $\sigma \downarrow 0$. Our method is different from theirs in the point that we define $\Pi_0, dx_\tau/N$ directly and treat both the Feynman integral and the Wiener integral on the same level.

2. The mathematical meaning of $\Pi_0, dx_\tau/N$

What Feynman had in mind for $\Pi_0, dx_\tau$ must be a uniform measure on $R(0,t)$. Rigorously speaking, this measure does not exist. Therefore, we should define it as an ideal limit of a sequence of measures on $R(0,t)$. In order to be able to compute the integral (1.3) or (1.5), the approximating measures should be concentrated on the set $L_x$ of all $X = (x_\tau, 0 < \tau \leq t) \in R(0,t)$ satisfying

(L.1) $x_\tau$ is absolutely continuous in $\tau$,
(L.2) $x_\tau = dx_\tau/d\tau \in L^2(0, t]$,
(L.3) $\lim_{\tau \downarrow 0} x_\tau = x$.

We shall now construct a sequence of probability measures $\{P_n\}$ on $L_x$ whose ideal limit is the uniform distribution on $R(0,t)$. Let $\rho(\tau, \sigma)$, with $\sigma, \tau \in (0, t]$, be strictly positive definite and continuous; for example, $\rho(\tau, \sigma) = \exp (-|\tau - \sigma|)$. Let $\xi(\omega), \omega \in \Omega(B, P)$, be a Gaussian process with

$$(2.1) \quad E(\xi_\tau) = 0, \quad E(\xi_\tau \xi_\sigma) = \rho(\tau, \sigma).$$

It is well known that such a Gaussian process exists. Since the continuity of $\rho(\tau, \sigma)$ implies the continuity of $\xi_\tau$ in the mean, there exists a measurable version [2] of $\xi_\tau$. Denote that version with the same symbol $\xi_\tau$.

Noting that

$$(2.2) \quad E \left( \int_0^t \xi_\tau^2 \, d\tau \right) = \int_0^t \rho(\tau, \tau) \, d\tau < +\infty,$$

we can see that

$$(2.3) \quad P \left\{ \int_0^t \xi_\tau^2 \, d\tau < +\infty \right\} = 1.$$
Put

$$x^{(n)}_\tau = x + n \int_0^\tau \xi \, d\theta, \quad 0 < \tau \leq t.$$  

Then \(x^{(n)}_\tau\), for \(0 < \tau \leq t\), is also a Gaussian process with

$$E[x^{(n)}_\tau] = x,$$

$$E[(x^{(n)}_\tau - x)(x^{(n)}_\tau - x)] = n^2 \int_0^\tau \int_0^\tau \rho(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2.$$  

Denote by \(P^{(n)}\) the probability distribution of the sample function \(X^{(n)}\) of the process \(x^{(n)}_\tau\), with \(0 < \tau \leq t\). Then \(P^{(n)}\) is concentrated on \(L_s\) and any finite dimensional marginal distribution of \(P^{(n)}\), say over coordinates \(\tau_1, \tau_2, \ldots, \tau_m\), is Gaussian with the density

$$\frac{b^{1/2}}{(2\pi n^2)^{m/2}} \exp \left[ -\frac{1}{2n^2} \sum_{i,j=1}^m b_{ij} (x_i - x)(x_j - x) \right],$$

where the matrix \((b_{ij})\) is the inverse of the matrix \((v_{ij})\) with

$$v_{ij} = \int_0^\tau \int_0^\tau \rho(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2, \quad i, j = 1, 2, \ldots, m$$

and \(b\) is the determinant of \((b_{ij})\). The existence of \((b_{ij})\), that is, the nonvanishing of the determinant of \((v_{ij})\) results from the assumption that \(\rho(\tau, \sigma)\) is strictly positive definite.

Since the Gaussian distribution (2.6) tends to a uniform distribution on the \(m\)-space in the sense that, for any almost periodic function \(f(x_1, x_2, \ldots, x_m)\),

$$\int \cdots \int f(x_1, \cdots, x_m) \frac{b^{1/2}}{(2\pi n^2)^{m/2}} \exp \left[ -\frac{1}{2n^2} \sum_{i,j=1}^m b_{ij} (x_i - x)(x_j - x) \right] \, dx_1 \cdots \, dx_m$$

tends to the Bohr mean \(\mathfrak{M}(f)\) of \(f\) as \(n \to \infty\), it is reasonable to say that \(P^{(n)}\), for \(n = 1, 2, \ldots\) approximates the uniform distribution on \(R^{(0,1)}\) and that \(\prod_x \, dx\) is an ideal limit of this sequence.

\(N\) must be also an ideal limit of a sequence of numbers \(\{N_n\}\) such that \(P^{(n)}/N_n\) tends to \(\prod_x \, dx/N\) in some sense.

Keeping these heuristic considerations in mind, we shall give a mathematical meaning to \(\prod_x \, dx/N\), that is, to the linear functional \(I(F) = \int F(X) \prod_x \, dx/N\).

There are many ways of defining this functional in accordance with the choice of the sequence \(\{N_n\}\). We shall express \(I(F)\) as \(I(F, N_n)\) referring to the sequence \(\{N_n\}\).

**Definition.**

$$I(F, N_n) = \lim_{n \to \infty} \frac{1}{N_n} \int_{L_n} F(X) P^{(n)}(dX).$$

The domain \(\mathfrak{D}(N_n)\) of this functional \(I(F, N_n)\) is the set of all \(F\) for which the limit in (2.9) exists and is finite.
Fixing \( \{N_n\} \), we shall write \( \mathcal{D}(N_n) \) as \( \mathcal{D} \) and \( I(F, N_n) \) as

\[
\frac{1}{N} \int_{\mathcal{D}} F(X) \prod_{\tau} dx_{\tau}.
\]

We shall mention three interesting cases.

(i) **Uniform integral.** \( N_n = 1 \), with \( n = 1, 2, \cdots \). If \( F(X) \) is of the form \( f(x_1, x_2, \cdots, x_m) \) with an almost periodic function \( f(x_1, x_2, \cdots, x_m) \), then \( F \in \mathcal{D} \) and

\[
\frac{1}{N} \int_{\mathcal{D}} F(X) \prod_{\tau} dx_{\tau} = \Re(f),
\]

where \( \Re(f) \) is the Bohr mean of \( f \).

(ii) **Wiener integral.** \( N_n = 1/(1 + n^2 \lambda_n)^{1/2} \), \( n = 1, 2, \cdots \), where \( \lambda_n \) will be defined in the next section. We shall discuss the Wiener integral in section 4.

(iii) **Feynman integral.** \( N_n = 1/(1 + n^2 \lambda_n/h\hbar)^{1/2} \), with \( n = 1, 2, \cdots \), with the same \( \lambda_n \) as in (ii). This will be discussed in section 5.

3. **Orthogonalization method**

In the following sections we shall be faced with the integrals of the type

\[
I = \int_{\Omega} G(X) P^{(\tau)}(dX).
\]

Recalling that \( P^{(\tau)}_n \) was defined as the probability distribution of the sample path \( X^{(\tau)} \) of the process

\[
x^{(\tau)}(\omega) = x + n \int_{\theta}^{\tau} \xi_{\tau}(\omega) d\theta
\]

introduced in section 2, the integral \( I \) can be expressed as the mean value of \( G[X^{(\tau)}(\omega)] \) on \( \Omega(\mathcal{B}, P) \)

\[
I = \int_{\Omega} G[X^{(\tau)}(\omega)] P(d\omega) = \mathbb{E}[G[X^{(\tau)}]].
\]

To compute this, we shall use the usual **orthogonalization method.** The idea is as follows. Let \( T \) denote the operator from \( L^2(0, t) \) into itself.

\[
(T\eta)(\tau) = \int_{0}^{\tau} \rho(\tau, \sigma)\eta(\sigma) d\sigma.
\]

Then \( T \) is a strictly positive-definite compact operator. Therefore \( T \) has positive eigenvalues \( \{\lambda_\tau\} \) whose eigenfunctions \( \{\eta_\tau\} \) constitute a complete orthonormal system in \( L^2(0, t) \).

Now put

\[
a_\tau(\omega) = \langle \xi, \eta_\tau \rangle = \int_{0}^{\tau} \xi_{\tau}(\omega)\eta(\tau) d\tau;
\]

this inner product is well defined, thanks to (2.3). Then \( \{a_\tau\} \) is a Gaussian system, since \( \xi \) is a Gaussian process. Equation (2.1) implies that
\( E(a_\nu a_\mu) = \int_0^t \int_0^t \rho(\tau, \sigma) \eta_\nu(\tau) \eta_\mu(\sigma) \, d\tau \, d\sigma. \)

Therefore \( a_\nu \), with \( \nu = 1, 2, \cdots \), are independent and each \( a_\nu \) is Gaussian with mean 0 and variance \( \lambda_\nu \). Since we have

\[
\sum_\nu \lambda_\nu = \sum_\nu E(a_\nu^2) = E(\sum_\nu a_\nu^2) = E\left( \int_0^t \xi_\nu^2 \, d\tau \right) = \int_0^t \rho(\tau, \tau) \, d\tau,
\]

the continuity of \( \rho(\tau, \sigma) \) implies

\( \sum_\nu \lambda_\nu < +\infty; \)

this fact will be useful in the following sections.

Noting that

\[
\xi_\nu(\omega) = \sum_\nu a_\nu(\omega) \eta_\nu(\tau)
\]

and that

\[
x_\nu^{(n)}(\omega) = x + n \int_0^\tau \xi_\nu(\omega) \, d\theta.
\]

we can express \( I \) in the form

\[
I = E\{H(a_1, a_2, \cdots)\}
\]

with some \( H \). Using the independence and the normality of \( \{a_\nu\} \), we can compute (3.11) more easily than the original form (3.1).

4. Wiener integral

We shall now discuss the integral (2.10) for

\[
N_n = \prod_\nu \left(1 + n^2 \lambda_\nu \right)^{1/2}, \quad n = 1, 2, \cdots,
\]

where \( \lambda_\nu \), with \( \nu = 1, 2, \cdots \), are the eigenvalues introduced in section 3.

We can verify easily the convergence of the infinite sums and infinite products appearing in this section by appealing to (3.8).

**Lemma 4.1.**

\[
Q_n^{(n)}(dX) = \frac{1}{N_n} \exp\left(-\int_0^t \frac{\xi_\nu^2}{2} \, d\tau \right) P_n^{(n)}(dX)
\]

is a probability distribution on \( L_\tau \).

**Proof.** Using the orthogonalization method, we get

\[
I_n = \int_{L_n} \exp\left(-\int_0^t \frac{\xi_\nu^2}{2} \, d\tau \right) P_n^{(n)}(dX)
\]

\[
= E\left[ \exp\left(-n^2 \int_0^t \xi_\nu^2 \, d\tau \right) \right]
\]

\[
= E\left[ \exp\left(-n^2 \sum_\nu \frac{a_\nu^2}{2} \right) \right].
\]
Noting that the $\alpha_\nu$ for $\nu = 1, 2, \cdots$, are independent, we have

\[ I_n = \prod_\nu E \left[ \exp \left( -\frac{n^2 \alpha_\nu^2}{2} \right) \right] \]

\[ = \prod_\nu \left( \frac{1}{(2\pi \lambda_\nu)^{1/2}} \int_{-\infty}^{+\infty} \exp \left( -\frac{\alpha^2}{2\lambda_\nu} - \frac{n^2 \alpha_\nu^2}{2} \right) d\alpha \right) \]

\[ = \prod_\nu \frac{1}{(1 + n^2 \lambda_\nu)^{1/2}} \]

\[ = N_n, \]

which proves our lemma.

**Lemma 4.2.** For any $g \in L^2(0, t]$, we have

\[ \int_{L_\nu} \exp \left[ i \int_0^t \tilde{x}_\nu g(\tau) \, d\tau \right] Q_n^{(2)} (dX) = \exp \left[ -\sum_\nu \frac{n^2 \lambda_\nu g^2_\nu}{2(\nu^2 \lambda_\nu + 1)} \right], \]

where $g_\nu = (g, \eta_\nu) = \int_0^t g(\tau) \eta_\nu(\tau) \, d\tau$, for $\nu = 1, 2, \cdots$.

**Proof.** By the same idea as in lemma 1, we have

\[ I_n = \int_{L_\nu} \exp \left[ i \int_0^t \tilde{x}_\nu g(\tau) \, d\tau \right] Q_n^{(2)} (dX) \]

\[ = \frac{1}{N_n} \int_{L_\nu} \exp \left[ i \int_0^t \tilde{x}_\nu g(\tau) \, d\tau - \int_0^t \tilde{x}_\nu^2 \, d\tau \right] P_n^{(2)} (dX) \]

\[ = \frac{1}{N_n} E \left[ \exp \left( i n \sum_\nu g_\nu \alpha_\nu - n^2 \sum_\nu \frac{\alpha_\nu^2}{2} \right) \right] \]

\[ = \frac{1}{N_n} \prod_\nu E \left[ \exp \left( i \eta_\nu g_\nu - n^2 \frac{\alpha_\nu^2}{2} \right) \right] \]

\[ = \frac{1}{N_n} \prod_\nu \left( \frac{1}{(2\pi \lambda_\nu)^{1/2}} \int_{-\infty}^{+\infty} \exp \left( \frac{-\alpha^2}{2\lambda_\nu} + i \eta_\nu \alpha - n^2 \frac{\alpha_\nu^2}{2} \right) d\alpha \right) \]

\[ = \frac{1}{N_n} \prod_\nu \frac{1}{(1 + n^2 \lambda_\nu)^{1/2}} \exp \left[ -\frac{n^2 \lambda_\nu g^2_\nu}{2(\nu^2 \lambda_\nu + 1)} \right], \]

which proves (4.5) by virtue of (4.1).

As an immediate result from lemma 4.2, we obtain the following lemma, noting that $\sum g^2_\nu = \int_0^t g(\tau)^2 \, d\tau$ and that $\int_0^t g(\tau) \, dB(\tau)$ is Gaussian distributed with mean 0 and variance $\int_0^t g(\tau)^2 \, d\tau$ for the Brownian motion $B(\tau)$.

**Lemma 4.3.** For any $g \in L^2(0, t)$,

\[ \exp \left[ i \int_0^t \tilde{x}_\nu g(\tau) \, d\tau - \int_0^t \tilde{x}_\nu^2 \, d\tau \right] \in D, \]
and

\[ \frac{1}{N} \int_{\Gamma_a} \exp \left[ i \int_0^t \dot{x}(\tau) \, d\tau - \int_0^t \frac{\dot{x}^2}{2} \, d\tau \right] \prod_{r} dx_r = \int_{\Gamma_a} \exp \left[ i \int_0^t g(\tau) \, dx_r \right] W_x(dX), \]

where \( W_x \) is the probability measure for the Brownian motion process starting at \( x \), namely the Wiener measure.

**Theorem 4.1.**

\[ \frac{1}{N} \exp \left( - \int_0^t \frac{\dot{x}^2}{2} \, d\tau \right) \prod_{r} dx_r = W_x(dX); \]

rigorously speaking, we have, for any continuous bounded tame function \( F(X) \),

\[ F(X) \exp \left( - \int_0^t \frac{\dot{x}^2}{2} \, d\tau \right) \in \mathbb{D}; \]

\[ \frac{1}{N} \int_{\Gamma_a} F(X) \exp \left( - \int_0^t \frac{\dot{x}^2}{2} \, d\tau \right) \prod_{r} x_r = \int_{\Gamma_a} F(X) W_x(dX). \]

A tame function is a function defined on an infinite dimensional space which depends only on a finite number of coordinates.

**Proof.** \( F(X) \) can be expressed as

\[ F(X) = f(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_m}), \quad 0 < \tau_1 < \cdots < \tau_m \leq t, \]

with a continuous bounded function \( f \) of \( m \) real variables. To obtain theorem 4.1, it is enough to prove that

\[ \lim_{n \to \infty} \int_{\Gamma_a} f(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_m}) Q_n^{(2)}(dX) = \int_{\Gamma_a} f(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_m}) W_x(dX). \]

Let \( Q_n^{(2)} \) and \( W_x \) denote the marginal distributions of \( Q_n^{(2)} \) and \( W_x \) over the coordinates \( \tau_1, \tau_2, \ldots, \tau_n \) respectively. Then

\[ I_n = \int \cdots \int \exp \left[ i(z_1 \alpha_1 + \cdots + z_m \alpha_m) \right] Q_n^{(2)}(d\alpha_1 \cdots d\alpha_m) \]

\[ = \int_{\Gamma_a} \exp \left[ i(z_1 x_{\tau_1} + \cdots + z_m x_{\tau_m}) \right] Q^{(2)}(dX) \]

\[ = \exp \left[ i(z_1 + \cdots + z_m) x \right] \int_{\Gamma_a} \exp \left[ i \int g(\tau) \dot{x}, d\tau \right] Q_n^{(2)}(dX), \]

where \( g(\tau) = \sum_{i=1}^m z_i \varphi_i(\tau) \) and \( \varphi_i(\tau) \) is the indicator function of the set \( (0, \tau_i) \).

Using lemma 4.3, we get

\[ I_n = \int_{\Gamma_a} \exp \left[ i(z_1 + \cdots + z_m) x + i \int g(\tau) \, dx_r \right] W_x(dX) \]

\[ = \int_{\Gamma_a} \exp \left[ i(z_1 x_{\tau_1} + \cdots + z_m x_{\tau_m}) \right] W_x(dX) \]

\[ = \int \cdots \int \exp \left[ i(z_1 \alpha_1 + \cdots + z_m \alpha_m) \right] W_x(d\alpha_1 d\alpha_2 \cdots d\alpha_n). \]

Therefore \( Q_n^{(2)} \to W_x \) in the weak sense as \( n \to \infty \), which implies (4.13).
Theorem 4.2. If \( f(x) : R^1 \to C \) is continuous and bounded, then

\[
\exp \left( - \int_0^t \frac{\dot{x}_t^2}{2} \, dt \right) f(x) \in D
\]

and

\[
u(t, x) = \frac{1}{N} \int_{\mathbb{R}} \exp \left( - \int_0^t \frac{\dot{x}_t^2}{2} \, dt \right) f(x) \prod_{\tau} dx, \tag{4.17}
\]

solves

\[
\frac{\partial \nu}{\partial t} = \frac{1}{2} \frac{\partial^2 \nu}{\partial x^2} - \frac{1}{2} \frac{x}{\partial x^2}, \quad u(0+, x) = f(x). \tag{4.18}
\]

Proof. Using the previous theorem, we have

\[
u(t, x) = \int_{\mathbb{R}} f(x) W_x(dX) = \int f(y) \frac{1}{(2\pi t)^{1/2}} \exp \left[ -\frac{(x - y)^2}{2t} \right] dy
\]

and this solves (4.18).

Theorem 4.3. (Feynman's version of Kac's theorem.) If \( f(x) \) and \( U(x) \) are continuous and bounded, then

\[
\exp \left\{ - \int_0^t \left[ \frac{\dot{x}_r^2}{2} + U(x_r) \right] \, dr \right\} f(x_t) \in D
\]

and

\[
\nu(t, x) = \frac{1}{N} \int_{\mathbb{R}} \exp \left\{ - \int_0^t \left[ \frac{\dot{x}_r^2}{2} + U(x_r) \right] \, dr \right\} f(x_t) \prod_{\tau} dx, \tag{4.21}
\]

solves

\[
\frac{\partial \nu}{\partial t} = \frac{1}{2} \frac{\partial^2 \nu}{\partial x^2} - U(x) \nu, \quad u(0+, x) = f(x). \tag{4.22}
\]

Proof. It is enough, by virtue of Kac's theorem, to prove that

\[
\lim_{n \to \infty} \frac{1}{N_n} \int_{L_n} \exp \left\{ - \int_0^t \left[ \frac{\dot{x}_r^2}{2} + U(x_r) \right] \, dr \right\} f(x_t) P_n(dX)
\]

\[= \int_{\mathbb{R}} \exp \left\{ - \int_0^t U(x_r) \, dr \right\} f(x_t) W_x(dX).
\]

Denoting the integrals in (4.23) by \( I_n \) and \( I \) respectively, we obtain

\[
I_n = \int_{L_n} \exp \left\{ - \int_0^t U(x_r) \, dr \right\} f(x_t) Q_{\alpha}^m(dX)
\]

= \( I_{nm} + R_{nm} \),

where

\[
I_{nm} = \sum_{r=1}^m \frac{(-1)^r}{r!} \int_0^t \cdots \int_0^t \int_{L_n} \prod_{\tau} U(x_{r}) \cdots U(x_{r}) Q_{\alpha}^m(dX) \, dr_1 \cdots dr_r,
\]

\[
|R_{nm}| \leq \frac{\ell m+1}{(m+1)!} \|U\|_{m+1} \|f\|_\infty \exp (\|U\|_\infty t), \tag{4.26}
\]

where \( \| \|_\infty \) means the uniform norm. Therefore \( I_{nm} \) tends to \( I_n \) uniformly in \( n \) as \( m \to \infty \). Using theorem 4.1, we have
\[ (4.27) \quad I_{nm} \to I_{\infty} \]

\[ = \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_0^t \cdots \int_0^t U(x_r) \cdots U(x_1) f(x_1) W(x) \, d\tau_1 \cdots d\tau_r \]

and it is easy to see that \( I_{\infty} \to I \) as \( n \to \infty \). Taking the uniform convergence of \( \lim_{m \to \infty} I_{nm} = I_n \) into account, we have

\[ (4.28) \quad \lim_{n \to \infty} I_n = \lim_{m \to \infty} \lim_{n \to \infty} I_{nm} = \lim_{m \to \infty} I_{nm} = \lim_{m \to \infty} I_{\infty} = I, \]

which completes our proof.

5. Feynman integral

In this section we shall discuss (2.10) for

\[ (5.1) \quad N_n = \frac{1}{\prod_r \left( 1 + \frac{n^2 \lambda_r}{\hbar^2} \right)^{1/2}}, \quad n = 1, 2, \ldots. \]

As in section 4, we can easily verify the convergence of the infinite sums and infinite products, by appealing to (3.8).

**Lemma 5.1.** If \( \text{Re}(b) > 0 \) and \( c \) is real, then

\[ (5.2) \quad \int_{-\infty}^{\infty} \exp(-b\alpha^2 + ic) \, d\alpha = \left( \frac{\pi}{b} \right)^{1/2} \exp \left( -\frac{c^2}{4b} \right). \]

**Proof.** This is true if \( b > 0 \). By analytic continuation, we can verify (5.2) for \( \text{Re}(b) > 0 \).

**Lemma 5.2.** If \( g(\tau) \in L^2(0, l) \), then

\[ (5.3) \quad \frac{1}{N_n} \int_{L_n} \exp \left[ \frac{i}{\hbar} \int_0^t \frac{\dot{\mathbf{x}}^2}{2} \, d\tau + i \int_0^t \mathbf{x}_r g(\tau) \, d\tau \right] P_n^{\omega}(d\mathbf{X}) = \exp \left[ -\sum_r \frac{n^2 \lambda_r a_r g_r}{2(n^2 \lambda_r + \hbar^2)} \right], \]

where

\[ (5.4) \quad g_r = (g, \eta_r) = \int_0^t g(\tau) \eta_r(\tau) \, d\tau. \]

**Proof.** We shall use the orthogonalization method introduced in section 3.

\[ (5.5) \quad I_n = \frac{1}{N_n} \int_{L_n} \exp \left[ \frac{i}{\hbar} \int_0^t \frac{\dot{\mathbf{x}}^2}{2} \, d\tau + i \int_0^t \mathbf{x}_r g(\tau) \, d\tau \right] P_n^{\omega}(d\mathbf{X}) \]

\[ = \frac{1}{N_n} E \left\{ \exp \left[ \frac{in^2}{2\hbar} \sum_r \left( a_r^2 + \text{ing}_r a_r \right) \right] \right\} \]

\[ = \frac{1}{N_n} E \left[ \exp \left( \frac{in^2}{2\hbar} a_r^2 + \text{ing}_r a_r \right) \right] \]

\[ = \frac{1}{N_n} \prod_r \int_{-\infty}^{\infty} \frac{1}{(2\pi \lambda_r)^{1/2}} \exp \left( -\frac{\alpha^2}{2\lambda_r} + \frac{in^2 \alpha^2}{2\hbar} + \text{ing}_r \alpha \right) \, d\alpha. \]
Using lemma 5.1 to evaluate this integral, we have

\[(5.6) \quad I_n = \exp \left[ -\sum \frac{n^3 \lambda_n h_i g_i^2}{2(n^3 \lambda_n + h_i)} \right]. \]

Noting that \( \sum g_i^2 = \int_0^t g(r)^2 \, dr \), we obtain, as an immediate result from lemma 5.2,

**Theorem 5.1.** If \( g(r) \in L^2(0, t] \), then

\[(5.7) \quad \exp \left[ \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr + i \int_0^t \hat{x}_r g(r) \, dr \right] \in \mathbb{D} \quad \text{and} \quad \frac{1}{N} \int_{\tau_r} \exp \left[ \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr + i \int_0^t \hat{x}_r g(r) \, dr \right] \prod \, dx_r = \exp \left[ -\hbar i \int_0^t \hat{g}^2(r) \, dr \right]. \]

**Theorem 5.2.** If the Fourier transform of \( \varphi(\hat{x}) \) is a continuous function with compact support, then

\[(5.9) \quad \exp \left( \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr \right) \varphi(x_r) \in \mathbb{D}, \]

and

\[(5.10) \quad \psi(t, x) = \frac{1}{N} \int_{\tau_r} \exp \left( \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr \right) \varphi(x_r) \prod \, dx_r \]

solves

\[(5.11) \quad \frac{\hbar \partial \psi}{i \partial t} = \frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial x^2} \quad \psi(0, x) = \varphi(x). \]

**Proof.** It is enough to prove that

\[(5.12) \quad I_n = \frac{1}{N_n} \int_{\Lambda_n} \exp \left( \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr \right) \varphi(x_r) P_n^{(\psi)}(dX) \]

tends to

\[(5.13) \quad \int \frac{1}{(2\pi \hbar i)^{1/2}} \exp \left[ -\frac{(x - \hat{y})^2}{2\hbar} \right] \varphi(y) \, dy. \]

Denoting the Fourier transform of \( \varphi(x) \) by \( \hat{\varphi}(\hat{x}) \) or \( \langle \varphi \rangle(x) \) as

\[(5.14) \quad \varphi(\hat{x}) = \langle \varphi \rangle(\hat{x}) = \int_{-\infty}^{\infty} \exp (2\pi i \hat{x} x) \varphi(x) \, dx, \]

we have

\[(5.15) \quad I_n = \frac{1}{N_n} \int_{\Lambda_n} \exp \left( \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr \right) \int_{-\infty}^{\infty} \exp \left( -2\pi i \hat{x}_r \varphi(\hat{x}) \right) \, d\hat{x} \quad P_n^{(\psi)}(dX) \]

\[= \frac{1}{N_n} \int_{-\infty}^{\infty} \varphi(\hat{x}) \exp \left( -2\pi i \hat{x}_r \varphi(\hat{x}) \right) \, d\hat{x} \int_{\Lambda_n} \exp \left( \frac{i}{\hbar} \int_0^t \frac{\hat{x}_r^2}{2} \, dr \right) \]

\[- 2\pi i \hat{x} \int_0^t \hat{x}_r \, dr \right) P_n^{(\psi)}(dX). \]
Putting \( g(t) = -2\pi \dot{x}t \), in lemma 5.2 to compute this integral over \( I_n \), we have

\[
I_n = \int_{-\infty}^{\infty} \varphi(\dot{x}) \exp \left[ -2\pi \dot{x} x - \sum \frac{n^2 \hbar \lambda_n \dot{x}^2}{2 (n^2 \lambda_n + \hbar \dot{x})} \right] dx.
\]

Recalling the assumption that \( \varphi(\dot{x}) \) has a compact support, and noting that

\[
\sum \dot{x}^2 = \int_0^t \dot{x}^2(\tau) d\tau = 4\pi^2 \dot{x}^2 t,
\]

we have

\[
\lim_{n \to \infty} I_n = \int_{-\infty}^{\infty} \varphi(\dot{x}) \exp \left( -2\pi \dot{x} x - 2\pi \hbar \dot{x}^2 t \right) d\dot{x}.
\]

Since the Fourier transform of

\[
N(x, \hbar \dot{x}) = \frac{1}{(2\pi \hbar \dot{x})^{1/2}} \exp \left( \frac{-x^2}{2\hbar \dot{x}} \right)
\]

in the Schwartz distribution sense [6] is \( \exp \left( -2\pi \hbar \dot{x} \dot{x} \right) \), we obtain

\[
\lim_{n \to \infty} I_n = \mathcal{F}^{-1}\{\mathcal{F} \varphi [N(\cdot, \hbar \dot{x})]\}
\]

namely

\[
\psi(t, x) = N(x, \hbar \dot{x}) \ast \varphi(x),
\]

which completes our proof.

**Theorem 5.3.** If the Fourier transform of \( \varphi(x) \) has compact support, then we have

\[
\exp \left[ \frac{i}{\hbar} \int_0^t \left( \frac{\dot{x}^2}{2} - x_\tau \right) d\tau \right] \varphi(x_0) \in \mathbb{D}
\]

and

\[
\psi(t, x) = \frac{1}{N} \int_{L_\tau} \exp \left[ \frac{i}{\hbar} \int_0^t \left( \frac{\dot{x}^2}{2} - x_\tau \right) d\tau \right] \varphi(x_\tau) \prod dx_\tau
\]

solves

\[
\frac{\hbar}{i} \frac{d\psi}{dt} = \frac{\hbar}{2} \frac{d^2 \psi}{dx^2} - x_0 \psi,
\]

\( \psi(0+, x) = \varphi(x) \).

**Proof.** Defining \( \varphi(\dot{x}) \) as in (5.14), we obtain

\[
I_n = \frac{1}{N_n} \int_{L_\tau} \exp \left[ \frac{i}{\hbar} \int_0^t \left( \frac{\dot{x}^2}{2} - x_\tau \right) d\tau \right] \varphi(x_\tau) P_n^{(\alpha)}(dX)
\]

\[
= \frac{1}{N_n} \int \varphi(\dot{x}) d\dot{x} \int_{L_\tau} \exp \left( \frac{i}{\hbar} \int_0^t \frac{\dot{x}^2}{2} d\tau - \frac{i}{\hbar} \int_0^t x_\tau d\tau - 2\pi i \dot{x} x_\tau \right) P_n^{(\alpha)}(dX)
\]

\[
= \frac{1}{N_n} \int \varphi(\dot{x}) \exp \left[ -2\pi i \left( \dot{x} + \frac{t}{2\pi \hbar} \right) x \right] d\dot{x} \int_{L_\tau} \exp \left[ \frac{i}{\hbar} \int_0^t \frac{\dot{x}^2}{2} d\tau \right.
\]

\[
+ \left. i \int_0^t g(\tau) d\tau \right] P_n^{(\alpha)}(dX),
\]
where \( g(\tau) = -(t - \tau)/\hbar - 2\pi \hat{x} \). Using lemma 5.2 to evaluate this integral over \( L_\pi \) and recalling that \( \phi \) has compact support to take the limit as \( n \to \infty \), we have

\[
(5.26) \quad \psi(t, \hat{x}) = \int \phi(\hat{x}) \exp \left[ -2i \left( \frac{\hat{x} + \frac{t}{2\pi \hbar}}{2} \right) x - \frac{\hbar i}{2} \int_0^t \left( \frac{t - \tau}{\hbar} - 2\pi \hat{x} \right)^2 d\tau \right] d\hat{x}
\]

\[
= \int \phi(\hat{x} - \frac{t}{2\pi \hbar}) \exp \left[ -2\pi i \hat{x} x - \frac{\hbar i}{2} \int_0^t \left( \frac{\tau}{\hbar} - 2\pi \hat{x} \right)^2 d\tau \right] d\hat{x}
\]

\[
= \int \phi(\hat{x} - \frac{t}{2\pi \hbar}) \exp \left[ -2\pi i \hat{x} x - \frac{\hbar^2 i}{6} \left( \frac{-2\pi \hat{x} + \frac{t}{\hbar}}{\hbar} \right)^3 - (-2\pi \hat{x})^3 \right] d\hat{x}.
\]

Thus we get

\[
(5.27) \quad \hat{\psi}(t, \hat{x}) = \mathcal{F}[\psi(t, \cdot)]
\]

\[
= \phi(\hat{x} - \frac{t}{2\pi \hbar}) \exp \left[ -\frac{\hbar^2 i}{6} \left( \frac{-2\pi \hat{x} + \frac{t}{\hbar}}{\hbar} \right)^3 - (-2\pi \hat{x})^3 \right].
\]

Simple computation shows that \( \hat{\psi}(t, \hat{x}) \) satisfies

\[
(5.28) \quad \frac{\hbar}{i} \frac{\partial \hat{\psi}}{\partial t} = \frac{\hbar^2}{2} (-2\pi i \hat{x})^2 \hat{\psi} - \frac{1}{2\pi i} \frac{\partial \hat{\psi}}{\partial \hat{x}} \quad \phi(0+, \hat{x}) = \phi(\hat{x}).
\]

This implies (5.13).

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REFERENCES