PROCEEDINGS of the FIFTH BERKELEY SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY

Held at the Statistical Laboratory
University of California
June 21—July 18, 1965
and
December 27, 1965—January 7, 1966

with the support of
University of California
National Science Foundation
National Institutes of Health
Air Force Office of Scientific Research
Army Research Office
Office of Naval Research

VOLUME III

PHYSICAL SCIENCES

EDITED BY LUCIEN M. LE CAM AND JERZY NEYMAN

UNIVERSITY OF CALIFORNIA PRESS
BERKELEY AND LOS ANGELES
1967
THE SPECTRAL ANALYSIS OF
LINE PROCESSES

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1. The specification of line processes

In recent papers [2], [3], I have discussed the spectral analysis of point processes in one or more dimensions, showing that the degenerate character of such processes does not prevent spectral analysis techniques, already familiar with continuous processes being adapted to such processes. The question arises, somewhat analogously as in the case of spectral or other distribution functions themselves, whether other forms of degeneracy will be encountered in practice; and, if so, what procedures are possible. One class of process which does arise in various contexts is what I have termed a line process ([2], p. 295) in which the points of a point process are replaced, in two or more dimensions, by lines. The example given referred to a number of vehicles on a road, treated for simplicity as points in a one-dimensional continuum, and thus at any instant as a point process. If the points are considered at two instants of time we have a bivariate point process, but if the points are plotted continuously over time as another coordinate the process will consist of a number of lines. This example makes two things clear. First, the specification of the process is partly optional, for the same process is either a point process (in a coordinate x, say) developing in time, or a “static” two-dimensional line process in x and the time coordinate t. Such alternative representations are not exhaustive, for (as in dynamics) the velocity u could also be included if convenient as an additional coordinate, though of course this is not necessary, as u is always derivable in the other specifications. Second, the lines in the line process need not be straight, as when the vehicles are accelerating. Indeed, in any general mathematical specification the lines might not even possess tangents at any point, as in a collection of Brownian particles. We shall, however, for definiteness assume that derivatives exist, as in our example. Moreover, as in the case of point processes, only particular classes of processes can be statistically analyzed by standard techniques. In the case of point processes, spectral analysis requires stationarity (or the equivalent property in more than one dimension). When discussing the spectral analysis of line processes, we shall not only assume an appropriate stationarity property, but shall also for simplicity consider processes consisting merely of straight lines, though not necessarily of infinite extent. Such a process in two dimensions is sometimes useful as an idealized representation of the fibers in a sheet of paper.
It should be noted (Bartlett, [1], [4]) that, whether or not the lines are straight, the density functions for line processes corresponding to different representations will satisfy various relations. For example, with the first order densities
\begin{equation}
\begin{aligned}
f_x(x, t) &= E\{d_xN(x, t)\}/dx, \\
f_t(x, t) &= E\{d_tN(x, t)\}/dt,
\end{aligned}
\end{equation}
we have
\begin{equation}
\begin{aligned}
f_x &= \int f_x u \ du, \\
f_t &= \int f_t u \ du, \\
f_{t,u} &= \|u\|f_{x,u}.
\end{aligned}
\end{equation}
When \( u \geq 0 \) for all possible \( u \),
\begin{equation}
f_t = \int u f_{x,u} \ du = \overline{u(x)}f_x,
\end{equation}
say,
\begin{equation}
\overline{u(t)}f_t = \int u f_{t,u} \ du = \int u^2 f_{x,u} \ du \geq \overline{u(x)^2}f_x,
\end{equation}
whence \( \overline{u(t)} \geq \overline{u(x)} \), a result well known in the theory of traffic flow.

2. The spectra of line processes

In the case of point processes, their degeneracy implies that the corresponding spectral functions must be defined in a suitably extended sense. A similar extension will be necessary for processes which are strictly line processes, though the appropriate definition will depend on whether the lines are finite or infinite. Elsewhere I have shown (Bartlett, [4], §6.52) that a (straight) line process may conveniently be included as a degenerate example of the more general process
\begin{equation}
X(r) = \int \xi(s - r) \ dN(s),
\end{equation}
where \( N(s) \) is some point process and \( \{\xi(r)\} \) is a random function associated with each point event of \( N(s) \) with the point as origin. The \( \xi(r) \) are in general different realizations for each such point.

In the very special and purely random case of \( N(s) \) a Poisson process and \( \xi(r) \) zero except on an infinite line of random orientation, we find \( f(\omega) \) varies as \( 1/\omega \), where \( f(\omega) \) is the (unstandardized) spectrum of \( X(r) \) and \( \omega^2 = \omega_1^2 + \omega_2^2 \). It is possible that direct measurement of the spectra of line (or near line) processes may be feasible in certain contexts; but in the analysis of line processes by digital computation it seems convenient to make use of any alternative representations to transform such a line process first to a convenient point process, and then to analyze this point process. Let us list some theoretical examples.
LINE PROCESSES

(i) In the case of the purely random family of straight lines \(x \cos \theta + y \sin \theta = p\), the line process is equivalent to the Poisson point process in the infinite strip \(p\) from \(-\infty\) to \(\infty\), and \(\theta\) from 0 to \(\pi\) (Kendall and Moran [5]).

(ii) For finite lines it is possible to think in terms of formula (2.1). Thus, if the lines are of fixed length \(2\ell\), we may consider the two-dimensional point process of their centers, and an angle variable \(\theta\) from 0 to \(\pi\) representing slope. (In the case of finite “arrows,” that is, lines with directions, \(\theta\) would vary from 0 to \(2\pi\).)

(iii) For finite lines of variable length \(2L\), there will be an additional random variable \(L\) for each point. It is possible to think of \(L\), or a transformation of it, as introducing a further dimension to the point process; but in practice, as such a point process would not in general be stationary even if the original process were, it seems preferable to specify \(L\) merely as an ancillary variable. The same procedure could apply to a set of particles (or vehicles) whose positions \(x\) and velocities \(u\) were given at a single instant \(t\) (or \(t\) and \(u\) at a single position \(x\)), giving rise to a one-dimensional point process with ancillary variable.

An interesting feature of the point process representations in (i) and (ii) is that the point process is specified on a particular coordinate structure which will affect its spectral function. Unlike \(p\) in (i) or \(r\) in (ii), \(\theta\) is an angle variable with a Fourier series spectrum. The combined spectrum for \(p\) or \(r\) with \(\theta\) will consequently be coefficients associated with a Fourier series, each coefficient of which will have the form of a spectral function for \(p\) or \(r\). (If the line process were specified in three dimensions, the single angle variable \(\theta\) would be replaced by two angles \(\theta\) and \(\phi\) determining position on a unit sphere, with a corresponding series of coefficients associated with expansions in spherical harmonics (see, for example, Bartlett, [4], §6.53).

Another feature to notice is that the angle variable \(\theta\) will be uniform for a completely random line process, but this does not apply to some transformed variable such as the slope \(s = \tan \theta\), for which the density is

\[
f(s) = \frac{1}{\pi} \frac{1}{1 + s^2}.
\]

This raises the problem whether in some other example, such as the traffic situation with vehicle velocities, there is any advantage in transforming the ancillary variable to an angle variable by such a transformation as \(\tan^{-1} s\), or more generally \(\tan^{-1} (s - s_0)\). It might be worth exploring this possibility somewhat further, though the “nonstationarity” in general of the point process so extended, even if the transformation is carefully chosen, seems to make the use of spectral analysis less relevant with this device, as previously noted. It was felt that a simpler and more empirical incorporation of the ancillary variable in any spectral analysis was likely to be more informative, and the procedure adopted is discussed below.
3. The spectral analysis of line processes

The two numerical examples given for illustration will be
(a) an artificial, purely random, straight line process;
(b) a set of time instants at which vehicles passed a point on a road, together
with their velocities as values of an ancillary variable.

Example (a) was chosen as a line process for which the representation (i) above
was possible, as distinct from the more general representation (ii) which would
have meant a more complicated spectral analysis. Similarly, example (b), while
perhaps more immediately classifiable as a point rather than a line process, is
similar but simpler than the first point process representation mentioned in (iii)
for lines of variable length. However, the "periodogram" sums are defined below
for both one- or two-dimensional point process representations, with either one
angle variable \( \theta \) or one ancillary variable \( U \). In the case of an angle variable
we write

\[
J_\theta(\omega) = \sqrt{\frac{2}{n}} \sum r e^{i(\omega X_r + \theta)}
\]

where \( n \) is the number of points with (column) vector coordinates \( X_r \) for
\( r = 1, \ldots, n \) and \( \omega' \) the (row) vector frequency (so that in two dimensions
\( \omega X = \omega x_1 + \omega x_2 \)). In the case of an ancillary variable \( U \), we consider the
somewhat more empirical sum

\[
J_U(\omega) = \sqrt{\frac{2}{n}} \sum r e^{i\omega X_r} \delta U_r,
\]

where \( \delta U_r \) stands for \( U_r - U \), \( U \) being the observed mean. For large \( n \), the
sampling properties of \( J_U(\omega) \) will not be affected to the first order by the use
of \( U \) in place of the true mean \( E\{U\} \). It seems convenient to measure \( U \) from
its mean, so that \( J_U(\omega) \) is zero if \( U \) does not vary; and thus, \( J_U(\omega) \) is kept as
distinct as possible from the unmodified sum \( J(\omega) \) (or \( J_0(\omega) \) in (3.1) above).

Corresponding to equation (3.1), there will be a spectral function of the
general form \( f(\omega, s) = \alpha_s(\omega) \), for \( s = 0, 1, 2, \ldots \). For example, in the case of
\( X_r \) representing the centers of lines of constant length, with \( \theta \) measuring their
angle of direction (0 to \( \pi \)), the assumption of independent \( \theta \) gives

\[
\alpha_s(\omega) = 1 + f(\omega) E\{e^{is(\delta s - \delta_0)}\}
\]

say, where \( 1 + f(\omega) \) is the spectral function for \( X \) (standardized to unity for
random \( X \)) and \( \delta_{s,0} \) is zero for even \( s \neq 0 \), and 1 for \( s = 0 \). For odd \( s \),
\( \delta_{s,0} = 4/\pi^2 s^2 \). In the simplified \( p, \theta \) representation for purely random lines of
infinite length, we may for convenience take the range 0 to \( \infty \) for \( p \) and 0 to \( 2\pi \)
for \( \theta \) (instead of \( -\infty \) to \( \infty \) for \( p \) and 0 to \( \pi \) for \( \theta \)). In this case we then have
\( \delta_{s,0} = 0 \) for all integers \( s \) except \( s = 0 \).

In an alternative extreme nonrandom case where \( \theta \) is constant, we should
have \( \delta_{s,0} \) in (2.5) equal to unity whatever \( s \).
Consider next the sum in (2.4), which we replace in theoretical studies by

\[ J'_{U}(\omega) = \sqrt{\frac{2}{n}} \sum_{r} e^{i\omega x} \Delta U_r, \]

where \( \Delta U_r = U_r - E(U_r) \). Then for \( I'_{U}(\omega) \) where \( I_{U}(\omega) = J_{U}(\omega)J'_{U}(\omega) \), and so forth, we obtain

\[ E\{I'_{U}(\omega)\} = \frac{2}{n} \int \int e^{i\omega y} E\{dN(x)\Delta U(x)dN(x+y)\Delta U(x+z)\}, \]

where \( N(x) \) is the point process for \( X, \) \( U(x) \) is \( U_r \) at a point \( X_r \), for which \( dN(x) = 1 \), and the integration is over the sample region containing the \( X_r \). If \( \Delta U_r \) is independent of \( N(x) \), there is no contribution to the integral except at \( z = 0 \), and \( E\{I'_{U}(\omega)\} \rightarrow \lambda \sigma^2 \delta \) for all \( \omega \neq 0 \), where \( \sigma^2 = E\{(\Delta U_r)^2\} \) and \( E\{dN(x)\} = \lambda dx \). More generally, we shall write

\[ E\{\lambda e^{i\omega y} \Delta U(x)dN(x+y)\Delta U(x+z)\} = \{\lambda \sigma^2 \delta(z) + \mu(z)\} \ dx \ dz. \]

If we write

\[ E\{dN(x)dN(x+z)\} - \lambda^2 dx \ dz = \{\delta(z) + \mu(z)\} \ dx \ dz, \]

the second term on the right-hand side of equation (2.8) can rise to \( \sigma^2 \mu(z) dx \ dz \) in the extreme case where \( U_r \) is perfectly correlated with \( U \), for points \( X_r, X \) contributing to \( \mu(z) \).

In order to examine further possibilities, let us consider the more general extended process

\[ dM(x) = dN(x)[1 + \xi \Delta U(x)], \]

where \( \xi \) is an arbitrary (possibly complex-valued) coefficient. Then \( E\{dM(x)\} = \lambda dx \), and for the complete covariance density \( \nu(z) \) for \( dM(x) \), we obtain

\[ \lambda(1 + \xi^* \mu(z)) \delta(z) + \nu(z), \]

say, where

\[ \nu(z) \ dx \ dz = \mu(z) \ dx \ dz + \xi^* E\{\Delta U(x)dN(x)dN(x+z)\} + \xi E\{\Delta U(x+z)dN(x)dN(x+z)\} + \xi^* \mu(z) \ dx \ dz. \]

To demonstrate the nature of these functions in a particular one-dimensional clustering model, suppose \( \Delta U \) in a cluster is associated with cluster size; for example, with traffic data large clusters might well be associated with low velocities if overtaking were difficult. For definiteness suppose the relation is linear, so that

\[ E(\Delta U|r) = \rho[r - E(r)], \]

where \( r + 1 \) is the total cluster size; and suppose the residual \( \Delta U - E(\Delta U|r) \) is otherwise correlated to extent \( \rho \) within a cluster. We find (see Bartlett, [2], p. 266)

\[ \nu(z|r) = E\{\lambda_0[f_r(z) + 2f_{r-1}(z) + \cdots + rf_1(z)] [1 + \xi^* \Delta U \Delta U' + (\xi + \xi^*) \Delta U]r\}, \]
where $\Delta U$, $\Delta U'$ are different $\Delta U$ in the same cluster, $f_r(z)$ is the $r$th convolution density of the interval between consecutive vehicles in a cluster, and $\lambda_c$ is the average density of clusters. After substitution from (3.11) and

$$E\{\Delta U \Delta U'|r\} = \rho \nu_r + \beta^2[r - E(r)]^2,$$

where $\nu_r$ is the variance of $\Delta U$ given $r$, we finally obtain

$$v(z) = E_r\{X_c[f_r(z) + 2f_{r-1}(z) + \cdots + f_1(z)]$$

$$[1 + \xi*\{\rho \nu_r + \beta^2[r - E(r)]^2\} + \beta(\xi + \xi*)[r - E(r)]\}$$

where $E_r$ denotes averaging over $r$.

Two conclusions from this formula are

(i) if $\beta$ in this model is zero, then $\nu_r = 2\sigma_2^2$, and the second term in (3.6) becomes $\rho \sigma_2^2 \mu(z)$. In general, however, for $\beta \neq 0$, the relation of $\mu_\omega(z)$ to $\mu(z)$ is more complicated;

(ii) if $\beta = 0$, no cross-spectral density terms (coefficients of $\xi$ and $\xi*$) arise, but in general for $\beta \neq 0$ further information may be available from the cross-spectrum of $dN(x)$ and $\Delta U(x)dN(x)$. In particular, information on the sign of $\beta$ is only available from the cross-spectrum.

4. Analysis of example (a)

The data for the first example consisted of the 50 lines shown in figure 1, with $p$ from 0 to $\infty$ and $\theta$ from 0 to $2\pi$ coordinates given in table I. The latter

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\theta$</th>
<th>$p$</th>
<th>$\theta$</th>
<th>$p$</th>
<th>$\theta$</th>
<th>$p$</th>
<th>$\theta$</th>
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<th>$\theta$</th>
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<td>2.461</td>
<td>48.32</td>
<td>4.683</td>
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<td>5.60</td>
<td>0.217</td>
<td>28.76</td>
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</tr>
<tr>
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<td>5.771</td>
<td>19.39</td>
<td>2.244</td>
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<td>25.74</td>
<td>3.630</td>
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<td>64.52</td>
<td>1.856</td>
<td>66.72</td>
<td>3.074</td>
<td>41.70</td>
<td>3.879</td>
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<td>1.637</td>
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<td>0.392</td>
<td>12.99</td>
<td>4.820</td>
<td>7.69</td>
<td>5.280</td>
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<tr>
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<td>0.017</td>
<td>27.03</td>
<td>4.223</td>
<td>62.76</td>
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<td>11.92</td>
<td>1.307</td>
<td>56.70</td>
<td>5.731</td>
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</table>

values were obtained by calculating $\tan^{-1}(x/y)$ from a pair of independent normal variables $x$ and $y$ (Tracts for Computers, No. 25), and the former converted from uniformly distributed numbers in the range 0 to 100 (Tracts for Computers, No. 24) by dividing by $\sqrt{2}$, thus ensuring that the 50 lines intersected a circle of radius $50\sqrt{2}$, and hence most of them a square of side 50 (two did not, but were retained in the analysis).
The values of $I_s(\omega_p) = J_s(\omega_p)J_s^*(\omega_p)$ were computed for $\omega_p = 2\pi p/50$, for $p = 1, \cdots, 100$ and $s = -5$ to 5. Individual values are not reproduced, but the frequency tables for each $s$ are summarized in table II, and the cumulative totals in steps of five at a time are given for each $s$ in table III. The distributions in table II do not appear unreasonable, apart perhaps from rather more large values in the row for $s = +3$ than would be expected. However, the overall average of 2.10 is near to the theoretical average of 2, and the variation of the averages for the different rows gives a $\chi^2$ of 17.86 with 10 d.f., which does not reach significance at the $P = 0.05$ level, namely, 18.31.

Figure 1
Fifty random straight lines, example (a).
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TABLE II

Frequency Tables for $I_s(\omega_p)$

<table>
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<th>3</th>
<th>4</th>
<th>5</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
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<th>Avg.</th>
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<td>27</td>
<td>8</td>
<td>8</td>
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<td>1</td>
<td>1</td>
<td>--</td>
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TABLE III

Cumulative Totals for $I_s(\omega_p)$

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5. Analysis of example (b)

The traffic data for the second example were kindly supplied to me by the National Road Research Institute, Stockholm, and consisted of the time instants in seconds of vehicles passing a fixed point in the northbound direction on a two lane road (E4) between Stockholm and Uppsala on September 16, 1961. Velocity
measurements were only measured approximately in 10 km/hr group intervals. This may preclude a very accurate study of spacing-speed relations, but should be adequate for the type of spectral analysis described above. The entire series was quite extensive, consisting of 1215 observations, in which five velocities were missing. A series of 320 complete observations was chosen (the maximum available was 325). The data are not reproduced here, but a graph of the first twelve vehicle times and velocities is shown in figure 2. The results obtained from this set were checked from another set of 320 observations, containing only

| TABLE IV |
| Block Totals of 16 for $H_p = \overline{U}I(\omega_p)$ and $H'_p = I^U_0(\omega_p)$ |

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<th>$p$</th>
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<tr>
<td>305-320</td>
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</table>
Values of $H_p = U^2 I(\omega_p)$ summed over blocks of 16 (1st series: continuous line; 2nd series: dotted line). The expected ultimate values are indicated by arrows.

In addition to the $J_U(\omega_p)$ of equation (3.2), a more standard point-spectrum analysis was made from $J(\omega_p)$, or rather from $UJ(\omega_p)$, so that in addition to $I_U(\omega_p)$ values were available of $U^2 I(\omega_p)$. The range of $p$ taken was from 1 to 320,
and block totals of 16 were recorded. For the first series, the value of $U$ is, in units of 5 km/hr, 14.77, so that the expected value of a block total of 16 in such units is $14.77^2 \times 2 \times 16 = 6661$ on the null hypothesis. The corresponding

\[ \text{Figure 4} \]

Values of $H'_p = I_U(\omega_p)$ summed over blocks of 16 (1st series: continuous line; 2nd series: dotted line). The expected ultimate values are indicated by arrows.

value on a random hypothesis for totals of $I_U(\omega_p)$ is $32\sigma^2$, estimated to be in the same units $32 \times 6.87 = 219.8$. The corresponding expected values for the second sum are $32 \times 14.84^2 = 7047$ and $32 \times 5.80 = 185.6$. The actual values obtained are given in table IV and figures 3 and 4. The significance of the rise
Average of $H_p$ for both series (continuous line) with similar average for $H'_p$ standardized to same ultimate level (dotted line).

$P = 0.05$. Significance levels (two sides) for any point are indicated.
near the origin is clear from figure 5 which shows $H_p$ averaged over the two series, with $H'_p$ *standardized to the same ultimate level*, and $P = 0.05$ significance levels (two sides, for each separate point).

6. Further discussion of results for example (b)

The results for $I(\omega_p)$ were expected to show a spectrum similar to the one depicted for traffic data by Bartlett ([2], figure 1), and both series broadly agree in this. In fact, while the average intervals between vehicles is somewhat lower (12.35 secs for the first series and 10.63 secs for the second, compared with 15.81 secs in the earlier example), the density has been standardized to unity; the previous theoretical model, as specified in my 1963 paper [2], would appear reasonably compatible with the present results. It is recalled that it embodied a clustering process, with a modified geometric distribution for cluster size (excluding the leading vehicle)

\[
(6.1) \quad p(r) = \begin{cases} 1 - c, & r = 0, \\ c\alpha^{-1}(1 - d), & r = 1, 2, \ldots, \end{cases}
\]

with $c = 1/9$, $\alpha = 2/3$. A dominant feature of the spectrum is the ratio of its value near $\omega = 0$ to its limiting value as $\omega$ increases, this being equal to

\[
(6.2) \quad m + \frac{\sigma^2}{m} = \frac{(1 - \alpha)^2 + c(3 - \alpha)}{(1 - \alpha)(1 + c - \alpha)}
\]

for the above model, where $m$ and $\sigma^2$ are the mean and variance of $r + 1$. It will be noticed that rather indirect information is provided on $c$ by formula (6.2).

The results for $I_U(\omega_p)$ are the more novel. The rise in $I_U(\omega_p)$ with $I(\omega_p)$, while somewhat more irregular, is present for both series, and is consistent with an anticipated correlation of velocities for vehicles in the same cluster. For the first series the values of $I_U(\omega_p)$ seem to remain a little high on average compared with the expected limit of 219.8 even for the larger values of $\omega$. In general, the relation of $I_U(\omega_p)$ to $I(\omega_p)$ can be complicated (see formula (3.14)); but any apparent persistence of $I_U(\omega_p)$ above its ultimate value for large $\omega$ is not repeated for the second series; and it was decided to consider, at least provisionally, the simple clustering model where $\beta$ is zero and velocity fluctuations within a cluster had constant correlation $\rho$. The individual differences of $I(\omega_p)$ or $I_U(\omega_p)$ from their ultimate values are of course subject to relatively large sampling error. However, the ratio $(H'_p/H_\infty - 1)/(H_p/H_\infty - 1)$ will be most accurate for large value of the denominator; and an overall estimate of $\rho$ was made by weighting by the square of the denominator. The values $H_p$, $H'_p$ were taken separately for the two series given in table IV, and the calculated values used for $H_\infty$, $H'_\infty$. The estimates of $\rho$ so obtained are 0.76 and 0.78, respectively, suggesting rather a high correlation within clusters.

Such an effect should be demonstrable in other ways. The correlation $\rho$ should
give rise to a detectable serial correlation between consecutive vehicle velocities, where

\[ \rho' = \rho(m - 1)/m. \]

With \( m = 4/3 \), \( \rho' = 0.19 \) when \( \rho = 0.76 \), and 0.20 when \( \rho = 0.78 \). The actual serial correlations were computed to be 0.26 from the first series and 0.29 from the second. The agreement seems fair: though it could be somewhat improved either (i) by increasing \( m \), or (ii) by increasing \( \rho \), or (iii) supposing that additional heterogeneity in traffic density may contribute to the observed serial correlations.

With the apparent high correlation of velocities within clusters another rough check on the consistency of the model is possible. Suppose for simplicity we consider the correlation to be near unity. Runs of identical velocities will then be assumed to arise from two contingencies: (i) clusters; (ii) fortuitous runs. If the velocity distribution with discrete categories has probabilities \( p_1, p_2, \ldots, p_k \), then runs of length \( s \) from a purely random series have probability

\[ p_1q_1 + p_2q_2 + \cdots + p_kq_k. \]

From the observed velocity distributions (for each series of 320 observations separately), the probabilities in (6.4) yield the calculated distributions of Table V,

| TABLE V |
|-----------------|-----------------|-----------------|
| **Distribution of Runs of Vehicles with Same Velocity** |
| \( s \) | 1st Series | 2nd Series |
| | \( p_s \) | Observed | \( p_s \) | Observed |
| 1 | 0.7541 | 126 | 0.7460 | 134 |
| 2 | 0.1763 | 44 | 0.1778 | 29 |
| 3 | 0.0488 | 11 | 0.0519 | 18 |
| 4 | 0.0146 | 4 | 0.0163 | 9 |
| 5 | 0.0044 | 1 | 0.0053 | 3 |
| 6 | 0.0012 | 3 | 0.0017 | 0 |
| 7 | 0.0004 | 0 | 0.0006 | 1 |
| 8 | 0.0001 | 2 | 0.0002 | 2 |
| 9+ | 0.0001 | 2 | 0.0002 | 0 |
| Total | 1.0000 | 193 | 1.0000 | 196 |
| Mean | 1.345 | 1.653 | 1.367 | 1.633 |

with the observed distributions shown for comparison. As the calculation is very rough, runs involving a single cluster of more than one for \( r > 0 \) are neglected (as well as the overlap of clusters). We then have the approximate equation for the first series, \( 1.345 + c/(1 - \alpha) = 1.653 \), the second term on the left being the expected increase in length of run due to clusters of more than one. With \( \alpha = 2/3 \), this gives \( c = 0.308/3 = 0.103 \), a value compatible with the
value 1/9 previously assessed [2]. This estimate, while rather crude, is of some interest in view of the comparative paucity of information on $c$ noted above. The corresponding figures for the second series are 1.367 (in place of 1.345), 1.633 (for 1.653), whence $c$ is $0.266/3$ (for the same $\alpha$), that is, 0.089.

It might be noted that the mean value of the velocity for the larger runs ($\geq 5$, say) is, in 5 km/hr units, 14.0 for the first series and 13.7 for the second, compared with an average over all vehicles of 14.8. This provides slight evidence of a $\beta < 0$ in (3.11), but hardly perhaps enough to justify fitting any more complicated model as represented by such formulae as (3.14). However, it was felt that calculation of the cross-spectrum would be of interest, and the results are described below. The relevant explicit evaluation of (3.14) for the clustering model is given in the appendix.

7. Calculation and discussion of the cross-spectrum

The cross-spectrum was conveniently computed by making use of the identity

$$U^2 + (\delta U)^2 - U^2 = -2U\langle \delta U \rangle,$$

where $\delta U = U - \bar{U}$. Thus, the spectrum of $UdN(x)$ was computed and hence, making use of (7.1), the cross-spectral function of $dN(x)$ and $\delta UdN(x)$. The results are given in Table VI (and figure 6), which gives $-G_p = -\bar{U}I_{12}(\omega_p)$, where

$$I_{12}(\omega_p) = A_1(\omega_p)A_2(\omega_p) + B_1(\omega_p)B_2(\omega_p),$$

the subscript 1 referring to $dN(x)$ and 2 to $\delta UdN(x)$. (Notice that under the assumptions for our model the imaginary terms in the cross-spectral function do not appear, so that it is sufficient to calculate $I_{12}(\omega_p)$ above.)

| TABLE VI |
| Block Totals of 16 for $-G_p$ |

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</table>

The most significant feature of $G_p$ is its negative value as $\omega \to 0$, implying the anticipated negative value for $\beta$. We shall confine our attention to the
Values of $-G_p = -UT_{12}(\omega_p)$ summed over blocks of 16
(1st series: continuous line; 2nd series: dotted line).

These values can only be appraised roughly from the graphs; but the following values were used:

1st series: $-G_0 = 2000$, $H_0 - H_\infty = 1 \frac{1}{2} \times 10^4 - 6661$, $H'_0 - H'_\infty = 600 - 220$;
2nd series: $-G_0 = 1500$, $H_0 - H_\infty = 1 \frac{3}{4} \times 10^4 - 7047$, $H'_0 - H'_\infty = 500 - 186$.

The estimate of $\beta$ from the first series then yields $-0.266$, and from the second, $-0.159$, with a mean for the two series of $-0.213$. As a direct check on the order of magnitude and significance of this estimate, we may utilize the mean
values of $U$ for the longer velocity runs noted at the end of the last section. These give (if we assume any such run all belongs to the same cluster) estimates of $\beta$ of $-0.141 \pm 0.170$ (1st series), $-0.243 \pm 0.210$ (2nd series), or a (weighted) mean of $-0.182 \pm 0.132$. However, the significance of this relation seems much more definite from the cross-spectrum (either from the overwhelming preponderance of negative values for $G_p$ at the lower end of the frequency range, or from their individual significance if the covariance $I_{1\nu}(\omega_p)$ is converted to a correlation).

With the estimate of $\beta$ obtained from the cross-spectrum for each series, we may revise our estimates of the within-cluster velocity correlation. We now write this as

\[(7.3) \quad \rho_0 = \rho_1 (1 - \rho_1^2) + \rho_1^2,\]

where $\rho_1$ is the correlation corresponding to $\beta$. Using the theoretical value of \((14/3)^{1/2}\) for $\sigma_1^2$ when $c = 1/9$, $\alpha = 2/3$, we have $\rho$ estimated to be $-0.127$ (1st series) and $-0.082$ (2nd series). Making use of the expression for $H_0$ given in the appendix, we obtain estimates of $\rho$ (with $v = \sigma_1^2(1 - \rho_1^2)$) of 0.815 (1st series) and 0.879 (2nd series), or finally of $\rho_0$ of 0.818 (1st series) and 0.880 (2nd series). These estimates are likely to have less bias, but to contain more error fluctuations than the previous estimates assuming $\beta = 0$, namely, 0.76 and 0.78. It is perhaps worth noting that with these somewhat higher correlations the expected serial correlations for the velocities are 0.20 and 0.22, a little nearer the observed values.

The interpretation of the above spectral analysis of traffic data in terms of a clustering model is not of course unique or exhaustive. An alternative (and not necessarily incompatible) interpretation in terms of flow density relations will be discussed elsewhere.

I am very much indebted to Stig Edholm, Head of the Traffic Department, National Road Research Institute, Stockholm, for sending me the traffic data for the second example. I am also much indebted to David Walley for his invaluable help in providing the computer programs and arranging the computations for these “extended” spectral analyses.

\[
\begin{align*}
\diamond & \quad \diamond \quad \diamond \quad \diamond \quad \diamond \\
\end{align*}
\]

APPENDIX

Evaluation of the spectrum of $dM(x)$ for the clustering model. Equation (3.14) has the form

\[(A.1) \quad E_r\{(A + Br + Cr^2)(f_r + 2f_{r-1} + \cdots + rf_1)\}.
\]

If we write $L(\psi)$ for the Laplace transform of $f_1$, we have for

\[(A.2) \quad \int_{-\infty}^{\infty} e^{-i\omega \psi(z)} \, dz\]
the expression

\[ G(-i\omega) + G(i\omega), \]

where \( G(\psi) \) is evaluated (if \( \nu = \nu, \) constant; otherwise the term in \( \rho \) is modified) as

\[ A \{ LE(r) + L^2E'(r-1) + L^3E'(r-2) + \cdots \} + B \{ LE(r^2) + L^2E'(r(r-1)) + L^3E'(r^2-r-2) + \cdots \} + C \{ LE(r^3) + L^2E'(r^2-r-2) + \cdots \}, \]

\( E' \) denoting expectation over all nonnegative values. Now

\[ E'(r(r-s)) = E'^{(r-s)^2} + sE'(r-s), \]

\[ E'(r^2(r-s)) = E'(r^2) + 2sE'(r-s) + s^2E'(r-s). \]

Further, for the modified geometric distribution,

\[ E'^{(r-s)^2} = \alpha E'(r^2), \quad E'(r^3) = \alpha E'(r^2), \]

\[ E(r^2) = \frac{c(1 + \alpha^2)}{(1 - \alpha)^2}, \quad E(r^3) = \frac{c(1 + 4\alpha + \alpha^2)}{(1 - \alpha)^3} \]

\[ G = \frac{AcL}{(1 - \alpha)(1 - \alpha L)} \frac{BcL(1 + \alpha - 2\alpha^2 L)}{(1 - \alpha)^2(1 - \alpha L)^2} \]

\[ + Cc(L)(1 + 4\alpha + \alpha^2)(1 - \alpha L)^2 + 2\alpha L(1 - \alpha L)(1 - \alpha)^2 + \alpha L(1 - \alpha)^2(1 + \alpha L) \]

where further

\[ A = \lambda c \left\{ 1 + \xi_1^*p_1 + \beta^2 c \xi_1^* \left( \frac{1 - \alpha}{1 - \alpha} \right)^2 \right\}, \]

\[ B = -\lambda c \left\{ \frac{2\beta^2 \xi_1^* c}{1 - \alpha} - \beta (\xi + \xi^*) \right\}, \]

\[ C = \lambda c \beta^2 \xi_1^*. \]

Rearranging terms, we may write this finally as

\[ \lambda c(1 + \xi^* p_1)cL \]

\[ \frac{(1 - \alpha)(1 - \alpha L)}{(1 - \alpha)^2(1 - \alpha L)} \left\{ c_2 - \frac{2c(1 + \alpha - 2\alpha^2 L)}{1 - \alpha L} \right\} \]

\[ + \frac{\lambda c \beta^2 \xi_1^* cL}{(1 - \alpha)^2(1 - \alpha L)} \left\{ \frac{c_2 - 2c(1 + \alpha - 2\alpha^2 L)}{1 - \alpha L} \right\} \]

\[ + \frac{\beta \lambda c(\xi + \xi^*) cL}{(1 - \alpha)^2(1 - \alpha L)} \left\{ \frac{1 + \alpha - 2\alpha^2 L}{1 - \alpha L} - c \right\}. \]

It is of interest to examine the relative values at \( \omega = 0 \) \( (L = 1) \) for the particular case \( c = 1/9, \alpha = 2/3. \) We obtain, with \( \lambda_c = 3\lambda = 3/4, \)

\[ \frac{3}{4}(1 + \xi^* p_1) + 46^5 \beta^2 \xi_1^* + 5\beta (\xi + \xi^*). \]
REFERENCES


