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U N C L A S S I F I E D

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OPTIMUM LOCATION OF HYPERDOP, DOVAP, AND RADAR STATIONS

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FOREWORD

The investigation reported herein was begun while the author was in the Test Department carrying out studies of data reduction methods. After the author's transfer to the Research Department, this work continued intermittently at low priority until the report was completed.

The work on this project was supported by funds for technical services.

This report was reviewed for technical accuracy by E. A. Fay, F. S. Howell, and L. E. Ward.

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ABSTRACT

Optimum locations were found for dovap and radar-ranging systems. The locations of ground stations given are those which, for a fixed missile position, minimize the effect of the geometry of the configuration on the precision of the determination of missile position. Certain optimum locations are also given for hyperdop, which are not usable on a ballistics test range.

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INTRODUCTION

Optimum locations were found for dovap and radar-ranging systems in the investigation reported here, extending considerably the work done in Ref. 1 and 2. The investigation was concerned solely with the mathematical problem as defined in these references. The locations of ground installations given are those which, for a fixed missile position, minimize the effect of the geometry of the configuration on the precision of the determination of missile position.

Basic results and background of the optimum-location problem are given in Ref. 1, 2, and 3 and are, therefore, used without discussion. As in the references, the volume of the ellipsoid of concentration is used in this report as a measure of precision. The results of Ref. 1 and 2 that the geometry of the configuration has an effect on precision quite separate from other factors are also used. It is evident from these results that only the angles between the several rays from ground installations to missile position are involved in the effect of the geometry, when the assumptions given and discussed in Ref. 1 and 2 are satisfied and when the stations are given equal weight factors (P_i or P_{ij}). These assumptions are made and these conclusions used throughout this report. The volume of the ellipsoid of concentration is proportional to $\sigma^3/\sqrt{\Delta}$, where σ is the common standard deviation of the observations and where Δ is a certain determinant determined by these angles. An optimum location, then, is one which yields an absolute maximum of Δ for a fixed number of ground stations.

The expressions given in Ref. 1 and 2 for the determinant for each of the three types of instrumentation are easily seen to involve only the direction cosines of the rays from ground station to "approximate missile position" and not the slant ranges from station to this position. Since the sum of squares of the members of a set of direction cosines is always exactly unity, Δ is always bounded and has a maximum. It is convenient in the present report to arrange these direction cosines in the 3-row matrix $X = (x_{ij})$, in which the jth column is the set of direction cosines of the ray from the jth ground station to the missile, for $j = 1, 2, \cdots, n$. For each of the types of instrumentation, it is shown below that there is a matrix A of constants, determined by the number of stations and the type of instrumentation, and such that $\Delta = |XAX'|$, where X' is the transpose of X.

The results obtained in the present investigation are first summarized and then derived in the separate sections below devoted to the respective types of instrumentation. The treatment of the radar case follows the treatment of Ref. 2 in not making use of azimuth and elevation information. (When Ref. 2 was prepared, the equipment available did not yield adequate information of this kind.)

Optimum locations for any number of stations are given for radar and dovap. Certain locations are shown to be singular for hyperdop in that the data-reduction methods of Ref. 1 fail, while the only optimum locations found for hyperdop are not usable on a ballistics test range. Numerical results showing the effect on accuracy of the number of stations and the variation in accuracy as the missile follows a trajectory through the optimum are given for radar and dovap. In conclusion, the report offers some general discussions on certain results found.

Appendix A gives mathematical proofs of certain theorems relating to determinants of the form $|XAX'|$. Two of these theorems are essential in the treatments of the three types of instrumentation:

THEOREM 1. *Rigid rotations in space of any configuration leave* Δ *unchanged.*

THEOREM 2. *The determinant* Δ *is never greater than the product of its main diagonal elements and is equal to this product if and only if the latter is zero or all the off-diagonal elements are zero.*

THE RADAR CASE

Summary

The known optimum locations in the radar case may be arranged for convenience in two series, both of which involve circular cones with the missile at the vertex. Because of Theorem 1, these cones may be considered initially to have vertical axes.

The first and more important series has a fixed element-axis angle of arc cos $1/\sqrt{3} \simeq 54° 44'$, and the stations are on elements of the cone. A case for five stations is illustrated in Fig. 1. The station-missile rays may be (1) equally spaced around the cone or (2) divided into groups of not less than three stations each with the stations of a group equally spaced and the groups arbitrarily oriented with respect to each other. These are the only spacings in this series for $n \leq 6$ stations: other spacings are possible, however, for larger values of n (shown in Appendix B).

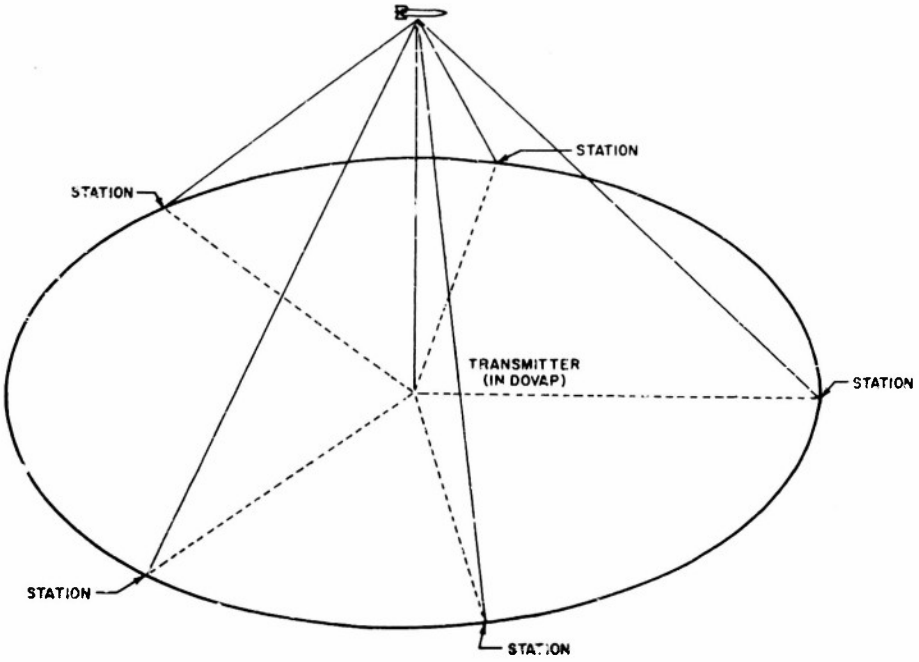
The second series has some stations on the axis of the cone. A construction in this series is possible only if the number, s, of stations on the axis is not greater than $n/3$, where n is the total number of stations. If $r = n - s$, the element-axis angle of the cone is arc cos $\sqrt{(1/r)[(n/3) - s]}$. The r stations not on the axis are on elements of the cone, spaced about it according to the same rules as used for the first series.

In addition to Theorem 1, another rule for deriving further optimum locations from a known optimum location is available in the radar case. The rule is that changing the sense of a station-missile ray leaves Δ unchanged.

These results are derived in detail below.

Derivation

The first result proved in this section is the rule stated above for deriving further optimum locations. This result is easily derived from the expressions for



RANGE PLANE

FIG. 1. Typical Configuration of Stations and Missile.

the elements of Δ. Such expressions are given on page 3 of Ref. 2 in terms of quantities α_1, β_1, and γ_1, which are the direction cosines x_{1i}, x_{2i}, and x_{3i} of the present report. Thus, expression 9 on page 3 of Ref. 2 is simply $\Delta = XX'$ (A is the unit matrix), and the element in the αth row and βth column of Δ is

$$\sum_{i=1}^{n} x_{\alpha i} x_{\beta i}$$

Evidently, this element is unchanged if x_{1i}, x_{2i}, and x_{3i} are replaced by $-x_{1i}$, $-x_{2i}$, and $-x_{3i}$, respectively; hence, the rule is obtained.

The second result proved is that locations of the first series are optimum. To show this, note that, when the cone axis is vertical, the vertical components x_{2i} are all equal and have the value $1/\sqrt{3}$.

With the stations in the range plane and the station-missile rays equally spaced around the cone, the stations are then equally spaced in the range plane around a circle whose center is directly below the missile. The pairs (x_{1i}, x_{3i}) are direction numbers in the range plane of the projections into this plane of the station-missile rays. Since $x_{1i}^2 + x_{3i}^2 = 1 - x_{2i}^2 = 2/3$, $x'_{1i} = x_{1i}\sqrt{3/2}$ and

$x'_{3i} = x_{3i}\sqrt{3/2}$ are direction cosines in the range plane of these projections. The complex numbers $z_i = x'_{1i} + x'_{3i}\sqrt{-1}$ lie on the unit circle in the complex plane and are equally spaced about this circle; hence, they are the nth roots of some complex number $\exp(\theta\sqrt{-1})$ whose absolute value is unity. Thus, $z_1, \cdots, z_n$ are the roots of the equation

(1) $$z^n - \exp(\theta\sqrt{-1}) = 0 \qquad (n \geq 3)$$

However, it is well known for any polynomial of the form

$$\zeta^n + a_1\zeta^{n-1} + a_2\zeta^{n-2} + \cdots + a_n$$

with zeros $\zeta_1, \cdots, \zeta_n$ that the relations

(2) $$\sum \zeta_i = -a_1$$
$$\sum_{i<j} \zeta_i\zeta_j = a_2$$

hold. From the first of these, since $a_1 = 0$ in the polynomial in Eq. 1,

$$\sum (x_{1i} + x_{3i}\sqrt{-1}) = 0 \quad \text{or} \quad \sum x_{1i} = 0 \quad \text{and} \quad \sum x_{3i} = 0$$

and it follows easily that

(3) $$\sum x_{1i}x_{2i} = (1/\sqrt{3}) \sum x_{1i} = 0 \quad \text{and} \quad \sum x_{3i}x_{2i} = (1/\sqrt{3}) \sum x_{3i} = 0$$

From the second of relations 2, since $a_2 = 0$ in Eq. 1,

$$\sum_{i<j} z_iz_j = (1/2)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} z_iz_j - \sum_{j=1}^{n} z_j^2\right) = -(1/2)\sum_{j=1}^{n} z_j^2 = 0$$

where the second step follows from $\Sigma z_j = 0$. However,

$$\sum_{j=1}^{n} z_j^2 = \sum_{j=1}^{n} [(x'_{1j})^2 - (x'_{3j})^2] + 2\sqrt{-1}\sum_{j=1}^{n} x'_{1j}x'_{3j}$$

so that

(4) $$\sum x_{1j}x_{3j} = 0$$

and

(5) $$\sum x_{1j}^2 = \sum x_{3j}^2$$

Relations 3 and 4 show that the off-diagonal elements of $|XX'|$ vanish so that Δ is equal to the product of its main diagonal elements. If these are

$$S_\alpha = \sum_{i=1}^{n} x_{\alpha i}^2 \qquad (\alpha = 1, 2, 3)$$

then, using $x_{1i}^2 + x_{2i}^2 + x_{3i}^2 = 1$,

(6) $$S_1 + S_2 + S_3 = n$$

and, since $x_{2i} = 1/\sqrt{3}$, $S_2 = n/3$ and, from Eq. 5, $S_1 = S_3$ also. This means that $\Delta = S_1S_2S_3 = (n/3)^3$. On the other hand, the partial derivatives of $Q = S_1S_2(n - S_1 - S_2)$ are

$$\frac{\partial Q}{\partial S_1} = S_2(n - 2S_1 - S_2)$$

$$\frac{\partial Q}{\partial S_2} = S_1(n - S_1 - 2S_2)$$

The obvious nontrivial solution of the equations setting these derivatives equal to 0 is $S_1 = S_2 = n/3$. Thus, $(n/3)^3$ is the absolute maximum of the product $S_1S_2S_3$, which Δ never exceeds by Theorem 2. The configurations of the first series for which the stations are equally spaced around the cone thus yield the maximum value of Δ and, hence, are optimum.

For the configurations of the first series in which the stations are grouped into r groups, the argument can be repeated in its entirety except that the polynomial of Eq. 1 becomes

$$[z^{n_1} - \exp(\theta_1\sqrt{-1})][z^{n_2} - \exp(\theta_2\sqrt{-1})] \cdots [z^{n_r} - \exp(\theta_r\sqrt{-1})]$$

where each $n_k \geq 3$ and $a_1 = a_2 = 0$ as before. The other locations not specified above follow from further results involving complex polynomials given in Appendix B.

The final result proved in this section is that locations of the second series are optimum. In this case, the matrix A takes the form

$$\begin{pmatrix} x'_{11}\sqrt{1-y^2} & x'_{12}\sqrt{1-y^2} & \cdots & x'_{1r}\sqrt{1-y^2} & 0 \cdots 0 \\ y & y & \cdots & y & 1 \cdots 1 \\ x'_{31}\sqrt{1-y^2} & x'_{32}\sqrt{1-y^2} & \cdots & x'_{3r}\sqrt{1-y^2} & 0 \cdots 0 \end{pmatrix}$$

where y is the elevation component common to all stations on the cone, where the last s columns of X apply to the stations on the cone axis, and where x'_{1i} and x'_{3i} may be taken as the real and imaginary parts of complex numbers z_i of unit modulus as above. The off-diagonal elements of Δ will now vanish for complex numbers z_i, subject to the same restrictions (with n replaced by r) as used for the first series. Moreover, since it has been shown that the main diagonal elements of Δ must equal $n/3$ for an absolute maximum, y must be chosen so that $ry^2 + s = n/3$ or $y = \sqrt{(n/3 - s)/r}$. Now $\Sigma_{i=1}(x'_{1i})^2 = r/2 = \Sigma_{i=1}(x'_{3i})^2$ so that the other main diagonal elements of Δ have the value

$$\frac{r}{2}(1 - y^2) = \frac{r}{2}\left[1 - \frac{1}{r}\left(\frac{n}{3} - s\right)\right]$$

$$= \frac{r+s}{2} - \frac{n}{6} = \frac{n}{3}$$

Discussion

If the locations are to be usable on a ballistics test range, the second rule for deriving further optimum locations must be used with discretion. Changing the sense of a station-missile ray evidently puts the station initially on the upper nappe of the cone. It is possible for several such changes to be made in such a way that the result cannot be rotated so as to have all stations in the range plane and the missile above it.

If a configuration has been modified by this rule and successfully rotated to place the stations in the range plane, the stations will always lie on a hyper-

bola in the range plane, with some stations on each branch. In a general way, the final result may be expected to have some stations near the range line and some quite far off-range, so that locations obtained in this way are considered less useful for a ballistics test range. In fact, practical consideration need hardly be given to locations other than those of the first series.

THE DOVAP CASE

Summary

As in the radar case, optimum locations in the dovap case may be specified in terms of a circular cone with the missile at the vertex (Fig. 1). The distinctive transmitter-missile ray in dovap is the axis of the cone, for which the element-axis angle is arc cos $1/3 \simeq 70° 32'$. The receiver-missile rays are elements of the cone and are distributed about it in the same way as in the radar case: (1) equally spaced around the cone or (2) for $n \geq 6$, divided into groups of not less than three receivers, with the members of each group equally spaced and the groups arbitrarily oriented with respect to each other. Other spacings indicated in Appendix B apply to dovap also.

In addition, the conclusion can be reached in the dovap case that all optimum locations are members of the above system. This is shown below following the proof that the locations stated are optimum.

Derivation

For the dovap case, it is convenient to suppose the matrix X to have $n + 1$ columns. Each of the first n columns is a set of direction cosines of a receiver-missile ray, and the last column is the set of the direction cosines of the transmitter-missile ray. From the discussion of the dovap case in Ref. 1, it is evident that the quantities b_{ij} (defined on page 9 of this reference) are sums of direction cosines. Thus, the matrix (b_{ij}) can be obtained by multiplying X by the matrix P on the right where

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and where P has $n + 1$ rows and n columns. Then the determinant $\Delta = |(b_{ij})(b_{ij})'|$ of Ref. 1 becomes $|XPP'X'| = |XAX'|$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

has $n + 1$ rows and columns, and is symmetric and nonnegative. If x_α denotes the αth row of X, then the element in the αth row and βth column of $\Delta = |XAX'|$ is

$$x_\alpha A x'_\beta = \sum_{i=1}^{n} x_{\alpha i} x_{\beta i} + \sum_{i=1}^{n} (x_{\alpha i} x_{\beta,n+1} + x_{\alpha,n+1} x_{\beta i}) + n x_{\alpha,n+1} x_{\beta,n+1}$$

By Theorem 1, the discussion may be concerned, without loss of generality, with the case in which the transmitter-missile ray is vertical: $(x_{1,n+1}, x_{2,n+1}, x_{3,n+1}) = (0, 1, 0)$. In this case,

$$\begin{aligned} x_1 A x'_1 &= S_1 = \sum x_{1i}^2 \\ x_2 A x'_2 &= S_2 = \sum (1 + x_{2i})^2 \\ x_3 A x'_3 &= S_3 = \sum x_{3i}^2 \\ x_1 A x'_2 &= x_2 A x'_1 = \sum (1 + x_{2i}) x_{1i} \\ x_1 A x'_3 &= x_3 A x'_1 = \sum x_{1i} x_{3i} \\ x_2 A x'_3 &= x_3 A x'_2 = \sum (1 + x_{2i}) x_{3i} \end{aligned}$$

where the sums, here and below, are over the range $i = 1, \cdots, n$.

The proof that the locations given in the summary of this section are optimum proceeds somewhat indirectly. It begins by finding the maximum of the product $S_1 S_2 S_3$ of the main diagonal elements of Δ, which is never exceeded by Δ itself according to Theorem 2. Since $x_{1i}^2 + x_{2i}^2 + x_{3i}^2 = 1$, this product may be written

$$S_1 S_2 S_3 = Q = \left(\sum x_{1i}^2\right)\left[\sum (1 + x_{2i})^2\right]\left[\sum (1 - x_{1i}^2 - x_{2i}^2)\right]$$

whence

$$\frac{\partial Q}{\partial x_{1k}} = 2x_{1k} S_2 S_3 - 2x_{1k} S_1 S_2$$

$$\frac{\partial Q}{\partial x_{2k}} = 2(1 + x_{2k}) S_1 S_3 - 2x_{2k} S_1 S_2$$

Since, obviously, each $S_\alpha > 0$ when Q is maximum, setting $\partial Q/\partial x_{1k} = 0$ implies $S_3 = S_1$, and setting $\partial Q/\partial x_{2k} = 0$ implies $x_{2k}(S_3 - S_2) = -S_3$. Now, if $S_3 = S_2$, this shows $S_3 = 0$; hence, for a maximum

$$x_{2k} = S_3/(S_2 - S_3) \tag{7}$$

for each $k = 1, 2, \cdots, n$. This means that all the x_{2k} are equal, say to some variable γ $(-1 \leq \gamma \leq 1)$, and all the receiver-missile rays lie in a right circular cone.

The equality of all the x_{2k}, which has to be satisfied in order for Q to be maximum, now simplifies the conditions of Theorem 2 which result in $\Delta = Q$. Thus, the off-diagonal elements are now zero if

$$\sum x_{1i} = \sum x_{3i} = 0 \quad \text{and} \quad \sum x_{1i}x_{3i} = 0 \tag{8}$$

In addition, it is shown above that $S_1 = S_3$ (i.e., that $\Sigma x_{1i}^2 = \Sigma x_{3i}^2$). This condition and Eq. 8 constitute the conditions satisfied by the optimum locations given in the radar case. Hence, the same distributions of rays around the cone give optimum locations in the dovap case. Moreover, since $x_{1i}^2 + x_{2i}^2 + x_{3i}^2 = 1$,

$$\sum x_{1i}^2 + \sum x_{3i}^2 = n(1 - y^2)$$

so that

$$S_1 = S_3 = (n/2)(1 - y^2)$$

and

$$\begin{aligned} Q = \Delta = S_1S_2S_3 &= [(n/2)(1 - y^2)][n(1 + y)^2][(n/2)(1 - y^2)] \\ &= (n^3/4)[(1 - y^2)(1 + y)]^2 \end{aligned}$$

and the value of Δ is a function of the cone angle alone when the receivers are distributed about the cone as specified.

The above relations also determine the cone angle as follows: From the relations on page 7,

$$S_1 + S_2 + S_3 = n + 2\sum x_{2i} + \sum (x_{1i}^2 + x_{2i}^2 + x_{3i}^2) = 2n(1 + y)$$

so that

$$S_2 = 2n(1 + y) - S_1 - S_3 = n(1 + y)^2$$

However, by Eq. 7,

$$\begin{aligned} S_2 &= S_3\left(1 + \frac{1}{y}\right) \\ &= n(1 + y)^2\,\frac{1 - y}{2y} \end{aligned}$$

since $S_3 = (n/2)(1 - y^2)$. Equating these expressions for S_2 yields $(1 - y)/2y = 1$ or $y = 1/3$. Thus, the cone angle is arc cos 1/3, and the maximum value of Δ is $256n^3/729$.

To prove, in conclusion, that all optimum locations are included in those discussed above, suppose a given configuration is, in fact, optimum. As above, it may be assumed in view of Theorem 1 that the transmitter-missile ray is vertical. Now the argument leading to Eq. 7 shows that the receiver-missile rays must lie in a circular cone and that $S_1 = S_3$. Further, it is shown above, for conical configurations, that the absolute maximum of Q is $256n^3/729$ and that $\Delta = Q$ when Eq. 8 are satisfied; hence, Eq. 8 are satisfied by the given configuration. Thus, if $n \leq 6$, the given configuration is one of the type specified in the summary of this section; if $n \geq 7$, other configurations are possible (shown in Appendix B).

THE HYPERDOP CASE

Summary

The work on the hyperdop case has produced two results of interest. The first is that, for any location of stations for which all station-missile rays are elements of a circular cone, the methods of data reduction of Ref. 1 fail. The second is a set of sufficient conditions for an optimum which can be satisfied for at least some n and for which the stations cannot be placed in the range plane with the missile above it.

Derivation

The first result follows, in fact, directly from the treatment of this case in Ref. 1. To show this, note that, by Theorem 1, this cone may be rotated so that its axis is vertical, resulting in the equality of all vertical components, x_{2i}. However, the x_{2i} are the quantities $(y_0 - Y_i)/D_i$ on page 4 of Ref. 1 so that the differences c_{ij2} vanish. Then $g_{rs} = 0$ when either $r = 2$ or $s = 2$ so that $\Delta = 0$, and the estimate of elevation of the missile cannot be found. Continuity considerations show, of course, that the accuracy of the position determination is poor for any configuration "approximately" conical.

The second result is obtained by using expressions for the elements of Δ. Evidently, the quantities c_{ijk} of Ref. 1 are the differences $x_{ki} - x_{kj}$ so that the element g_{rs} in the rth row and sth column of Δ is

$$\begin{aligned} g_{rs} &= \sum_{i<j} (x_{ri} - x_{rj})(x_{si} - x_{sj}) \\ &= \sum_{i<j} (x_{ri}x_{si} + x_{rj}x_{sj}) - \sum_{i<j} (x_{ri}x_{sj} + x_{rj}x_{si}) \\ &= (n-1) \sum_i x_{ri}x_{si} - \sum_{i \neq j} x_{ri}x_{sj} \\ &= n \sum_i x_{ri}x_{si} - \left(\sum_i x_{ri}\right)\left(\sum_i x_{si}\right) \end{aligned}$$

with $P_{ij} = 1$ and all pairs (i, j) with $i < j$ used. It is easily verified that the determinant $|g_{rs}|$ is of the form $|XAX'|$, where A is the symmetric, nonnegative matrix whose main diagonal elements are all $n - 1$, and whose off-diagonal elements are -1. Thus, the product of main diagonal elements of Δ is

$$Q = \left[n \sum x_{1i}^2 - \left(\sum x_{1i}\right)^2\right]\left[n \sum x_{2i}^2 - \left(\sum x_{2i}\right)^2\right]\left[n \sum x_{3i}^2 - \left(\sum x_{3i}\right)^2\right]$$

It is necessary to show that the partial derivatives of Q vanish at a maximum. This is done by use of the auxiliary function

$$\begin{aligned} Q^* &= \left[n \sum x_{1i}^2 - \left(\sum x_{1i}\right)^2\right]\left[n \sum x_{2i}^2 - \left(\sum x_{2i}\right)^2\right]\left[n \sum (1 - x_{1i}^2 - x_{2i}^2)\right] \\ &= S_1 S_2 S_3 \end{aligned}$$

in which no restriction is placed on the $2n$ variables x_{1i}, x_{2i}. From comparison of the expressions for Q and Q^*, it is evident that, when $x_{1i}^2 + x_{2i}^2 \leq 1$, $Q^* \geq Q$ for every choice of signs for $x_{3i} = \pm\sqrt{1 - x_{1i}^2 - x_{2i}^2}$, with $Q^* = Q$ when $\Sigma\, x_{3i} = 0$. Now,

Q^* is defined and has partial derivatives for all x_{1i}, x_{2i}; $Q^* < 0$ for some x_{1i}, x_{2i}; and $Q^* < 0$ if any $|x_{1i}|$ or $|x_{2i}|$ is sufficiently large. Hence, Q^* has a maximum at a finite point, and

$$\partial Q^*/\partial x_{1k} = \partial Q^*/\partial x_{2k} = 0$$

at this maximum for all k. However,

$$\partial Q^*/\partial x_{1k} = S_2\left[S_3\left(2nx_{1k} - 2\sum x_{1i}\right) - 2nS_1 x_{1k}\right]$$

which vanishes when $S_2 = 0$ or when $n(S_3 - S_1)x_{1k} = \Sigma\, x_{1i}$. Now, if $S_3 \neq S_1$, this last equality can only be satisfied if all x_{1k} are equal, which results in $\Delta = 0$, as is shown in the first paragraph of the derivation of this section. On the other hand, if $S_3 = S_1$, then $\Sigma\, x_{1i} = 0$, and a similar argument relative to $\partial Q^*/\partial x_{2k}$ shows that, for a maximum, $\Sigma\, x_{2i} = 0$ also. At a maximum, then, Q^* reduces to

$$n^3\left(\sum x_{1i}^2\right)\left(\sum x_{2i}^2\right)\left(n - \sum x_{1i}^2 - \sum x_{2i}^2\right)$$

From the discussion of the radar case, the absolute maximum of this quantity is $n^3(n/3)^3$, attained whenever $\Sigma\, x_{1i}^2 = \Sigma\, x_{2i}^2 = n/3$.

Thus, if a matrix X exists for which

$$\sum_i x_{\alpha i} = 0, \qquad \sum x_{\alpha i}^2 = n/3 \qquad (\alpha = 1, 2, 3)$$

and

$$\sum_i x_{\alpha i} x_{\beta i} = 0 \qquad (\alpha \neq \beta)$$

then, using Theorem 2, $\Delta = Q = Q^*$, and the absolute maximum of all of these is attained. The first three of these conditions show that some of the elements of each row of X are positive and some are negative, since $\Delta = 0$ if all elements of any row are zero; and it is easily verified that the same statement will hold after any rigid rotation. This means, of course, that some stations are above the missile and others are below it however the system is rotated.

Discussion

It is not known whether the sufficient conditions for an optimum of the previous discussion can be satisfied for all $n \geq 4$. It is easily verified, however, that configurations having the missile at the center and the stations along the rays to the vertices of a regular tetrahedron and a cube, respectively, satisfy these conditions. Obviously, configurations built up from tetrahedra and cubes in arbitrary relative orientation are optimum also.

NUMERICAL ILLUSTRATION

A number of numerical evaluations of the determinant were made for the radar and dovap cases in order to study their variations as the missile approached the optimum and as the number and arrangement of stations were changed. The same missile positions were used for both radar and dovap, including four points on a parabola passing through the launcher with vertex 50,000 feet down-range

and 20,000 feet in elevation. Except for one case, the configurations are optimum for the missile at the vertex, and they differ between the radar and dovap cases only in the necessarily larger circle in the range plane for the dovap receivers.

The coordinates of the missile position are given below, using x for down-range, y for elevation, and z for off-range.

	Missile position, ft			
Coordinate	*1*	*2*	*3*	*4*
x	50,000	45,000	30,000	5,000
y	20,000	19,800	16,800	3,800
z	0	0	0	0

Five arrangements of stations were used. The first had the minimum number (3) of stations; the second and third had 4 stations in different orientations with respect to the down-range axis; the fourth had 6 stations at the vertices of a regular hexagon; and the fifth had 3 stations in a nonoptimum configuration simulating a case of 4 stations with no record from one of them. The coordinates of these configurations are given in Table 1.

TABLE 1. CONFIGURATIONS OF STATIONS USED IN NUMERICAL CALCULATIONS

The dovap transmitter always has coordinates (50,000, 0, 0), and y is always zero.

Configuration	Coordinate, ft			
	Radar		Dovap	
	x	z	x	z
1. Equilateral triangle	78,284	0	106,569	0
	35,858	24,495	21,716	48,990
	35,858	−24,495	21,716	−48,990
2. Square with diag. on range line	78,284	0	106,569	0
	50,000	28,284	50,000	56,569
	21,716	0	−6,569	0
	50,000	−28,284	50,000	−56,569
3. Square with 2 sides parallel to range line	70,000	20,000	90,000	40,000
	30,000	20,000	10,000	40,000
	30,000	−20,000	10,000	−40,000
	70,000	−20,000	90,000	−40,000
4. Hexagon	78,284	0	106,569	0
	64,142	24,495	78,284	48,990
	35,858	24,495	21,716	48,990
	21,716	0	−6,568	0
	35,858	−24,495	21,716	−48,990
	64,142	−24,495	78,284	−48,990
5. Isosceles right triangle[a]	78,284	0	106,569	0
	50,000	28,284	50,000	56,569
	21,716	0	−6,569	0

[a]Simulating configuration 2 with one station missing.

TABLE 2. VALUES OF DETERMINANTS OBTAINED IN CALCULATIONS

Station configuration	Radar missile position				Dovap missile position			
	1	2	3	4	1	2	3	4
1	1.000	0.975	0.555	0.005	9.481	9.050	4.677	0.084
2	2.370	2.353	2.126	0.033	22.475	21.604	13.262	1.752
3	2.370	2.352	1.767	0.018	22.475	21.601	12.731	0.366
4	8.000	7.881	6.968	0.104	75.852	72.913	44.417	5.480
5	0.592	0.588	0.[illegible]2	0.008	5.619	5.389	3.201	0.376

The determinants were evaluated for every combination of missile position and station configuration. The values found are given in Table 2. The following remarks may be made relative to the values in this table:

1. The values do not permit a general comparison of accuracy between radar and dovap, since account is not taken of other sources of error.

2. Dovap consistently loses more accuracy than radar in missile positions 2 and 3 but less in position 4.

3. Of the two 4-station configurations (2 and 3), the one with 2 stations on the range line is better for both radar and dovap.

FURTHER OBSERVATIONS

The determinants Δ for optimum locations on cones with vertical axes have properties worthy of further discussion. The fact that the off-diagonal elements are always zero means that the three components of the fix are uncorrelated. This property simplifies further statistical inferences based on the data, such as estimates of velocity and acceleration. Again, the main diagonal elements of Δ remain in fixed ratio for all n in each type of instrumentation. These ratios show that (1) for radar, the optimum ellipsoid of concentration is always a sphere and (2) for dovap, the vertical axis of the ellipsoid is always one-half the other axes, which are always equal.

Another criterion of accuracy having strong intuitive appeal is the expected square of the radial error of the fix. The results of this report suggest that, for radar, the same locations are optimum by this criterion. For dovap, however, this criterion apparently leads to different locations for which the components of the fix may not be uncorrelated; nor is there good reason to suppose that the ellipsoid of concentration is spherical. Locations optimum by the alternative criterion should not be markedly different from those of this report, however, in the sense that regions of "good" fixes by the two criteria should overlap extensively.

Appendix A
PROOFS OF THEOREMS

The two theorems given in the Introduction and a third theorem of general character are proved in this appendix. Theorem 3 actually led to the discovery of optimum locations in the dovap case, although the results are more easily given otherwise.

THEOREM 1. *Rigid rotations in space of the system of rays whose direction cosines constitute X leave $\Delta = |XAX'|$ unchanged.*

PROOF. Let R be the matrix of the rotation so that RX is the matrix of direction cosines after the rotation. Then R is orthogonal, $|R| = 1$, and $|RXAX'R'| = |R|\ |XAX'|\ |R'| = |XAX'|$.

The proof of Theorem 2 requires a theorem of Hadamard. This is given on pages 219 and 220 of Ref. 4 as follows: Let $D = |q_{ij}|$ be an n-row determinant of real elements, and let $\Sigma_{j=1}^{n} q_{ij}^2 = p_i > 0$. Then the maximum of D^2 is $p_1 \cdots p_n$, and this maximum is attained if and only if

$$\sum_{k=1}^{n} q_{ik} q_{jk} = 0 \qquad (i \neq j)$$

THEOREM 2. *Let $X = (x_{ij})$ be a matrix of variables with m rows and n columns of rank m, and let $x_i = (x_{i1}, \cdots, x_{in})$ be the ith-row vector of X. If the matrix A is symmetric and positive semidefinite (i.e., all its principal minors are nonnegative), and if $x_i A x_i' > 0$ for $i = 1, \cdots, m$, then*

$$|XAX'| \leq (x_1 A X_1') \cdots (x_m A x_m')$$

where the equality sign holds if and only if $x_i A x_j' = 0$ when $i \neq j$.

PROOF. Let $B = (b_{ij})$ be a square matrix such that $BB' = A$, and set $XB = Y = (y_{ij})$. If $m < n$, let

$$\bar{Y} = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{m1} & \cdots & y_{mn} \\ z_{m+1,1} & \cdots & z_{m+1,n} \\ \cdots & \cdots & \cdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$$

where the z_{jk} are chosen as solutions of the homogeneous equations

$$\sum_{k=1}^{n} y_{ik} z_{jk} = 0 \qquad (i = 1, \cdots, m;\ j = m+1, \cdots, n)$$

which are orthogonal and normalized—that these exist may be seen from the theory of linear spaces—so that

$$\sum_{k=1}^{n} z_{ik}^2 = 1, \qquad \sum_{k=1}^{n} z_{ik} z_{jk} = 0 \qquad (i \neq j;\ i, j \geq m+1)$$

and let $\tilde{Y} = Y$ if $m = n$. Now

$$|\tilde{Y}|^2 = |\tilde{Y}\tilde{Y}'| = \begin{vmatrix} YY' & 0 \\ 0 & I_{n-m} \end{vmatrix} = |YY'|$$

On the other hand,

$$|\tilde{Y}|^2 \leq \left(\sum y_{1k}^2\right) \cdots \left(\sum y_{mk}^2\right)$$

by the Hadamard theorem, and the equality holds if and only if $\Sigma\, y_{ik} y_{jk} = 0$ for $i \neq j$. Generally,

$$\begin{aligned} \sum_k y_{ik} y_{jk} &= \sum_k \left(\sum_s x_{is} b_{sk}\right)\left(\sum_t x_{jt} b_{tk}\right) \\ &= \sum_{s,t} \left(\sum_k b_{sk} b_{tk}\right) x_{is} x_{jt} \\ &= \sum_{s,t} a_{st} x_{is} x_{jt} = x_i A x_j' \end{aligned}$$

Thus, for $i = j$,

$$|YY'| = |XAX'| \leq (x_1 A x_1') \cdots (x_m A x_m')$$

while the equality is seen to hold, by the Hadamard theorem, if and only if $x_i A x_j' = 0$ for $i \neq j$. This completes the proof.

Note. $|XAX'| = |\tilde{Y}|^2 \geq 0$.

Theorem 2, when applied to $\Delta = |XAX'|$, shows that

$$\Delta \leq Q = \left(\sum_{i,j} a_{ij} x_{1i} x_{1j}\right)\left(\sum_{i,j} a_{ij} x_{2i} x_{2j}\right)\left(\sum_{i,j} a_{ij} x_{3i} x_{3j}\right)$$

while $\Delta = Q\ (\neq 0)$ if and only if $\Sigma_{i,j}\, a_{ij} x_{\alpha i} x_{\beta j} = 0$ for $\alpha \neq \beta$.

Theorem 3, which is given below, simplifies the results obtained by applying the method of Lagrange to the problem of maximizing Q under the restrictions

$$x_{1i}^2 + x_{2i}^2 + x_{3i}^2 = 1 \tag{9}$$

The method yields, in addition to Eq. 9, the necessary conditions

$$\begin{aligned} &\sum_j a_{kj} x_{1j} + \lambda_k^{(1)} x_{1k} = 0 \\ &\sum_j a_{kj} x_{2j} + \lambda_k^{(2)} x_{2k} = 0 \\ &\sum_j a_{kj} x_{3j} + \lambda_k^{(3)} x_{3k} = 0 \end{aligned} \tag{10}$$

for $k = 1, \cdots, n$, where

$$\lambda_k^{(1)} = \lambda_k/S_2S_3, \qquad \lambda_k^{(2)} = \lambda_k/S_1S_3, \qquad \lambda_k^{(3)} = \lambda_k/S_1S_2$$

(11)

$$S_\alpha = \sum_{i,j} a_{ij}x_{\alpha i}x_{\alpha j} \qquad (\alpha = 1, 2, 3)$$

THEOREM 3. *Necessary and sufficient conditions that a matrix X constitute a solution of the system* 10 *subject to Eq.* 11 *are that there exist a matrix $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ and constants $\rho_i > 0$, $i = 1, 2, 3$, such that*

(12) $$(A + \rho_i\Lambda)x_i' = 0$$

and

(13) $$x_i\Lambda x_i' = -(1/\sqrt{\rho_1\rho_2\rho_3})$$

for $i = 1, 2, 3$, where x_i is the row vector $(x_{i1}, \cdots, x_{in})$.

PROOF. For the proof of the necessity of the conditions, if X is a solution of system 10, the matrix Λ and the constants ρ_i exist such that Eq. 12 is satisfied. Moreover,

$$\rho_1 = 1/S_2S_3, \qquad \rho_2 = 1/S_1S_3, \qquad \rho_3 = 1/S_1S_2$$

From these relations, it follows easily that

(14) $$S_1/\rho_1 = S_2/\rho_2 = S_3/\rho_3$$

It also follows from Eq. 12 that $x_i(A + \rho_i\Lambda)x_i' = 0$; that is,

(15) $$S_i = x_iAx_i' = -\rho_ix_i\Lambda x_i'$$

and substitution in Eq. 14 results in

(16) $$x_1\Lambda x_1' = x_2\Lambda x_2' = x_3\Lambda x_3'$$

Now the corresponding value of Q, say Q_0, is given by

$$Q_0 = (-\rho_1x_1\Lambda x_1')S_2S_3$$

whence, since $\rho_1S_2S_3 = 1$,

(17) $$Q_0 = -x_1\Lambda x_1'$$

But

$$\frac{S_1}{\rho_1}\left(\frac{S_2}{\rho_2}\right)\frac{S_3}{\rho_3} = \frac{Q_0}{\rho_1\rho_2\rho_3}$$
$$= (-x_1\Lambda x_1')(-x_2\Lambda x_2')(-x_3\Lambda x_3')$$
$$= Q_0^3$$

that is,

(18) $$Q_0 = 1/\sqrt{\rho_1\rho_2\rho_3}$$

and, using Eq. 16 and 17, it is seen that Eq. 13 is established.

For the proof of the sufficiency of the conditions, it follows from Eq. 13 that

$$(x_2\Lambda x_2')(x_3\Lambda x_3') = 1/\rho_1\rho_2\rho_3$$

whence

$$\rho_1\Lambda = \Lambda/(-\rho_2x_2\Lambda x_2')(-\rho_3x_3\Lambda x_3')$$

By Eq. 15, this relation is equivalent to

$$\rho_1 \dot{\Lambda} = \Lambda / S_2 S_3$$

and similar expressions hold for $\rho_2 \Lambda$ and $\rho_3 \Lambda$. Thus, with $\lambda_k^{(\alpha)} = \rho_\alpha \lambda_k$, Eq. 12 and 13 imply that X satisfies a system like the system 10 subject to Eq. 11, and the proof is complete.

Note. It follows from Eq. 9, 16, 17, and 18 that

$$\begin{aligned} 3Q_0 &= -\sum \lambda_i (x_{1i}^2 + x_{2i}^2 + x_{3i}^2) \\ &= -\sum \lambda_i = 3/\sqrt{\rho_1 \rho_2 \rho_3} \end{aligned}$$

Appendix B
METHODS OF CONSTRUCTING FURTHER CONFIGURATIONS FOR LARGE n

It is shown in the body of the report that necessary and sufficient conditions for a set of plane direction cosines (x_{1k}, x_{3k}) to determine an optimum location for either radar or dovap are

$$\begin{aligned} &\sum x_{1k} = 0 \qquad && \sum x_{3k} = 0 \\ &\sum x_{1k}^2 = n/2 \qquad && \sum x_{1k}x_{3k} = 0 \qquad (k = 1, \cdots, n) \end{aligned} \tag{19}$$

It is also shown that the monic complex polynomial whose zeros are

$$z_k = x_{1k} + x_{3k}\sqrt{-1},\ k = 1, \cdots, n,$$

is of the form

$$z^n + a_3 z^{n-3} + \cdots + a_{n-1}z + a_n \tag{20}$$

Conversely, it is clear that, for any complex polynomial of the form 20 with zeros of modulus unity, an optimum location can be constructed. As a consequence of the following theorem, the polynomial is seen to take a still more special form.

THEOREM 4. *If the complex polynomial*

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$$

with zeros $z_1, \cdots, z_n$ *is such that* $|z_k| = 1$ *for all* $k = 1, \cdots, n$, *then* $a_j = a_n \bar{a}_{n-j}$.

PROOF. As is well known,

$$\begin{aligned} a_j &= (-1)^j \sum_{k_1 < \cdots < k_j} z_{k_1} \cdots z_{k_j} \\ &= (-1)^j \sum_{k_1 < \cdots < k_j} 1/(\bar{z}_{k_1} \cdots \bar{z}_{k_j}) \end{aligned}$$

since $|z_k| = 1$. Hence,

$$a_j \bar{a}_n = (-1)^{j+n} \sum_{k_1 < \cdots < k_j} (\bar{z}_1 \bar{z}_2 \cdots \bar{z}_n)/(\bar{z}_{k_1} \cdots \bar{z}_{k_j})$$

The right member is now just $\bar{a}_{n-j}$, and, since $|a_n| = 1$, the theorem follows.

From this theorem, it follows that optimum locations are determined by complex polynomials of the form

$$z^n + a_3 z^{n-3} + \cdots + \bar{a}_3 z^3 + 1 \tag{21}$$

with zeros of unit modulus, where the specialization $a_n = 1$ merely rotates the zeros equally around the unit circle. The only polynomials of this form for $n = 3, 4, 5$ are $z^n + 1$, and the zeros of these are equally spaced on the unit

circle; hence, the only optimum locations of 3, 4, or 5 receivers have them equally spaced around the cone. For $n = 6$, the polynomial is quadratic in z^3; hence, each of its factors leads to 3 stations in an equilateral triangle, while the triangles may be arbitrarily oriented with respect to each other.

For larger values of n, various special methods have been used to find further optimum locations. For example, using the above theorem, if a polynomial of the form

$$z^r + b_2 z^{r-2} + \cdots + b_{r-2} z^2 + 1$$

has zeros of unit modulus, then the polynomial obtained from it by replacing z by z^s for $s \geq 2$ is a polynomial of the form 21 and leads to an optimum location for $n = rs$ stations. This method does not lead to further optimum locations for $r < 5$ or $s > 2$.

Another special method uses Eq. 19 in their equivalent trigonometric form

$$\sum \cos\theta_k = 0 \qquad \sum \sin\theta_k = 0$$
$$\sum \cos^2\theta_k = n/2 \qquad \sum \sin\theta_k \cos\theta_k = 0$$

where θ_k is the argument of the complex number z_k. This method has led to additional configurations for $n = 7$ in the special case $\theta_4 = -\theta_1$, $\theta_5 = -\theta_2$, $\theta_6 = -\theta_3$, and $\theta_7 = 0$. This assumption reduces the above equations to

$$t_1 + t_2 + t_3 = -\tfrac{1}{2}$$

$$t_1^2 + t_2^2 + t_3^2 = \tfrac{5}{4}$$

where $t_k = \cos\theta_k$, $k = 1, 2, 3$. These equations have the solutions

$$t_2 = -\tfrac{1}{4}\left(2t_1 + 1 - \sqrt{9 - 4t_1 - 12t_1^2}\right)$$

$$t_3 = -\tfrac{1}{4}\left(2t_1 + 1 + \sqrt{9 - 4t_1 - 12t_1^2}\right)$$

from which all locations may be obtained by choosing t_1 from the interval $\left[\frac{1}{2}, (2\sqrt{7} - 1)/6\right]$. Locations obtained in this way include the following:

$\left(t_1 = \frac{1}{2}\right)$. $\theta_1 = 60°$, $\theta_2 = 90°$, $\theta_3 = 180°$; an equilateral triangle with a vertex at -1 and a square with diagonal on the real axis

$t_1 = \cos(360°/7)$. A regular heptagon

$t_1 = \frac{1}{2}\sqrt{2}$. $\theta_1 = 45°$, $\theta_2 = 120°$, $\theta_3 = 135°$; another case of equilateral triangle and square

$t_1 = (2\sqrt{7} - 1)/6$. $\theta_1 = 44° 20'$, $\theta_2 = \theta_3 = 127° 25'$; an irregular pentagon with 2 stations at each of 2 vertices

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