AD No. 34595
ASTIA FILE COPY

ON THE POWER OF A ONE-SIDED TEST OF FIT AGAINST STOCHASTICALLY COMPARABLE ALTERNATIVES

By

Z. W. Birnbaum and Ernest M. Scheuer

University of Washington

Technical Report No. 15

June 18, 1954

Contract N8onr-520 Task Order II
Project Number NR-042-038

Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle, Washington

Reproduced

FROM LOW CONTRAST COPY.

THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

I INTRODUCTION

Let F be the continuous cumulative distribution function (c.d.f.) for the random variable (r.v.) X and F_n the empirical c.d.f. determined by the ordered sample $X_1, X_2, \ldots, X_n$ of X. F_n is defined by

$$(1.1) \qquad F_n(x) = \begin{cases} 0 & \text{for } x < X_1 \\ k/n & \text{for } X_k \leq x < X_{k+1};\ k=1,\ldots,n-1 \\ 1 & \text{for } x \geq X_n \end{cases}$$

It is known [1] that the probability $P\{F(x) \leq F_n(x) + \varepsilon, \text{ all } x\}$ is a function independent of F. We will use this function to test the hypothesis $F=H$ against the alternative $F=G$. The power of the test will be studied for alternatives G such that $G(x) \leq H(x)$ for all x and such that

$$\sup_{-\infty < x < -\infty} [H(x) - G(x)] = \delta, \text{ with pre-assigned } \delta > 0.$$

Alternatives of this kind will be called "stochastically comparable, at distance δ from H." For brevity's sake we shall refer to them as alternatives (A).

We assume throughout that $H \in (F)$, $G \in (F)$ where (F) is the set of all continuous strictly increasing c.d.f.'s.

We test $F=H$ against $F=G$ by the following procedure. To have a test of size α for sample size n we will use the value $\varepsilon_{n,\alpha}$ from Table 1 in [2], obtain an ordered sample $X_1, X_2, \ldots, X_n$ of X, determine the empirical c.d.f. of X and reject H if and only if the inequality

$$(1.2) \qquad H(x) < F_n(x) + \varepsilon_{n,\alpha}$$

fails to hold for all real x.

The power of this test is the complementary probability to

$$(1.3) \qquad P = P\left\{ H(x) < F_n(x) + \varepsilon_{n,\alpha}, \text{ for all } x \;\middle|\; G \right\}.$$

Since (1.2) is satisfied for all x if and only if

$$(1.4) \qquad H(X_i) < \frac{i-1}{n} + \varepsilon_{n,\alpha} \text{ for } i=1,\ldots,n$$

we have (writing hereafter ε for $\varepsilon_{n,\alpha}$ and noting that $H(x) \leq 1$ for all x)

$$P = P\left\{ H(X_i) < \min\left(\frac{i-1}{n} + \varepsilon, 1\right),\ i=1,\ldots,n \;\middle|\; G \right\}$$

$$= P\left\{ X_i < \min\left(H^{-1}\left(\frac{i-1}{n} + \varepsilon\right), H^{-1}(1)\right),\ i=1,\ldots,n \;\middle|\; G \right\}.$$

$$= P\left\{ G(X_i) < \min\left(GH^{-1}\left(\frac{i-1}{n} + \varepsilon\right), GH^{-1}(1)\right),\ i=1,\ldots,n \;\middle|\; G \right\}.$$

Define the function L by

$$(1.5) \qquad L(V) = \begin{cases} \lim\limits_{V \to 0+} GH^{-1}(V) & \text{for } V \leq 0 \\ GH^{-1}(V) & \text{for } 0 < V < 1 \\ \lim\limits_{V \to 1-} GH^{-1}(V) & \text{for } V \geq 1 \end{cases}$$

Recall that $U = G(X)$ has the rectangular distribution R in the unit interval when X has c.d.f. G. We conclude

$$(1.6) \qquad P = P\left\{ U_i < L\left(\frac{i-1}{n} + \varepsilon\right),\ i=1,\ldots,n \;\middle|\; R \right\}.$$

Clearly $U_1, U_2, \ldots, U_n$ is an ordered sample of the U's. Since the joint probability distribution of $(U_1, U_2, \ldots, U_n)$ is equal to n! for $0 \leq U_1 \leq U_2 \leq \ldots \leq U_n \leq 1$ and zero elsewhere, we have

$$(1.7) \qquad P = n! \int_0^{L(\varepsilon)} \int_{U_1}^{L(\frac{1}{n}+\varepsilon)} \cdots \int_{U_{n-1}}^{L(\frac{n-1}{n}+\varepsilon)} dU_n \ldots dU_2\, dU_1.$$

If $H(x) = G(x)$, then $L(V) = V$ for $0 < V < 1$ and (1.7) reduces to formula (3.3) in [2]. If H and G are given, it may be possible to evaluate P, and hence the power 1-P, from (1.7) by quadrature, or one may compute it by numerical integration. In the general case for alternatives (A_1) it is possible to derive from (1.7) inequalities for the power, as will be seen.

II ALTERNATIVES (a)

For given hypothesis H, we consider here alternatives G such that $G(x) \quad H(x)$ for all x and such that $\sup_{-\infty<x<\infty} [H(x)-G(x)] = \delta$ for pre-assigned $\delta > 0$. In view of the assumption that $H \in (F)$, $G \in (F)$, the supremum is actually attained, say at X_0. That is,

$$H(X_0)-G(X_0)=\delta.$$

For intuitive reasons one may expect that under these restrictions the power of our test will be close to its minimum when G is close to the function G* defined by

$$(2.1) \qquad G^*(x) = \begin{cases} H(x) & \text{for } x<H^{-1}(U_0) \\ U_0 & \text{for } H^{-1}(U_0)\leq x<x_0 \\ H(x) & \text{for } x\geq X_0 \end{cases}$$

where $U_0 = H(X_0)-\delta = G(X_0)$.

To verify this conjecture we consider

$$(2.2) \qquad L^*(V) = G^* H^{-1}(V) = \begin{cases} V & \text{for } 0<V<U_0 \\ U_0 & \text{for } U_0\leq V<V_0 \\ V & \text{for } V_0\leq V<1 \\ 1 & \text{for } V\geq 1 \end{cases}$$

where $V_0 = H(X_0)$.

Let

$$(2.3)\qquad \begin{aligned} j &= [n(U_o - \varepsilon)] \\ k &= [n(V_o - \varepsilon)] \\ \ell &= [n(1 - \varepsilon)] \end{aligned}$$

where $[A]$ = greatest integer less than A .

We want to show $L(V) \leq L^*(V)$ for all V. We note first that for $0 < V < 1$

$$(2.4)\qquad L(V) \leq V.$$

Suppose $X = H^{-1}(V)$. Then $V = H(x)$ and $G(x) = GH^{-1}(V) = L(V)$. By assumption $G(x) \leq H(x)$, therefore $L(V) \leq V$.

We note next that for $j+1 \leq m \leq k$ we have $\frac{m}{n} + \varepsilon \leq \frac{k}{n} + \varepsilon \leq V_o$. Therefore

$$(2.5)\qquad L(\tfrac{m}{n} + \varepsilon) \leq L(V_o) = U_o \qquad \text{for } j+1 \leq m \leq k.$$

Summing up

$$L(V) \leq V = L^*(V) \quad \text{for } 0 < V < U_o \text{ and } V_o < V < 1,$$

which means

$$(2.6)\qquad L(\tfrac{m}{n} + \varepsilon) \leq L^*(\tfrac{m}{n} + \varepsilon) \quad \text{for } 0 \leq m \leq j \text{ and } k+1 \leq m \leq \ell$$

Further, $L(V) \leq 1$ always, so

$$(2.7)\qquad L(\tfrac{m}{n} + \varepsilon) \leq 1 = L^*(\tfrac{m}{n} + \varepsilon) \quad \text{for } \ell + 1 \leq m.$$

Formulas (2.5), (2.6), (2.7) show us that replacing in (1.7) the function L by the function L^* in the upper limits of integration will

not decrease these limits. Hence

$$(2.8)\quad P \leq n! \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} \int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \int_{U_{\ell+1}}^{1} \cdots \int_{U_{n-1}}^{1}$$

$$dU_n \cdots dU_{\ell+2}\, dU_{\ell+1} \cdots dU_{k+2}\, dU_{k+1} \cdots dU_{j+2}\, dU_{j+1} \cdots dU_2\, dU_1 .$$

Denote the integral on the right by I. We proceed to evaluate I. By (2.1) of 2, we have

$$\int_{U_{\ell+1}}^{1} \cdots \int_{U_{n-1}}^{1} dU_n \cdots dU_{\ell+2} = \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}, \text{ so}$$

$$(2.9)\quad I = \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} \int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}$$

$$dU_{\ell+1} \cdots dU_{k+2}\, dU_{k+1} \cdots dU_{j+2}\, dU_{j+1} \cdots dU_2\, dU_1 .$$

Next we want to evaluate

$$(2.10)\quad \int_{U_{k+1}}^{\frac{k+1}{n}+\varepsilon} \cdots \int_{U_\ell}^{\frac{\ell}{n}+\varepsilon} \frac{(1-U_{\ell+1})^{n-\ell-1}}{(n-\ell-1)!}\, dU_{\ell+1} \cdots dU_{k-2}$$

Instead we evaluate a slightly more general expression. We shall have occasion to use it later in our discussion as well as now.

Let

$$(2.11)\quad T_j(c,d,\varepsilon) = \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon} (c-U_{j+1})^d\, dU_{j+1} \cdots dU_2\, dU_1$$

Then we have

LEMMA 1:

$$(2.12)\quad T_j(c,d,\varepsilon) = \frac{d!}{(d+j+1)!}\left[c^{d+j+1} - \varepsilon \sum_{t=0}^{j} \binom{d+j+1}{t} \left(c-\frac{t}{n}-\varepsilon\right)^{d+j+1-t} \left(\varepsilon+\frac{t}{n}\right)^{t-1}\right]$$

PROOF: By induction on j.

1) For $j=0$, trivial.

2) Assume the result true for all j, $1 \le j \le m$. Then

$$T_{m+1}(c,d,\varepsilon) = \int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}\int_{U_{m+1}}^{\frac{m+1}{n}+\varepsilon}(c-U_{m+2})^{d}\,dU_{m+2}\cdots dU_1$$

$$= \frac{1}{d+1}\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}(c-U_{m+1})^{d+1}\,dU_{m+1}\cdots dU_1$$

$$- \frac{\left(c-\frac{m+1}{n}-\varepsilon\right)^{d+1}}{d+1}\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_m}^{\frac{m}{n}+\varepsilon}dU_{m+1}\cdots dU_1$$

$$= \frac{1}{d+1}T_m(c,d+1,\varepsilon) - \frac{\left(c-\frac{m+1}{n}-\varepsilon\right)^{d+1}}{d+1}\cdot\frac{\varepsilon}{(m+1)!}\left(\varepsilon+\frac{m+1}{n}\right)^{m}$$

(See (2.2) of [2].)

$$T_{m+1}(c,d,\varepsilon) = \frac{1}{d+1} \cdot \frac{(d+1)!}{[(d+1)+m+1]!} \Bigg[c^{(d+1)+m+1} -$$

$$\varepsilon \sum_{t=0}^{m} \binom{(d+1)+m+1}{t} \left(c - \frac{t}{n}\varepsilon\right)^{(d+1)+m+1-t} \left(\varepsilon + \frac{t}{n}\right)^{t-1}$$

$$-\varepsilon \frac{[(d+1)+m+1]!}{(t+1)!(m+1)!} \left(c - \frac{m+1}{n} - \varepsilon\right)^{d+1} \left(\varepsilon + \frac{m+1}{n}\right)^{m} \Bigg]$$

$$= \frac{d!}{[d+(m+1)+1]!} \Bigg[c^{d+(m+1)+1} - \varepsilon \sum_{t=0}^{m+1} \binom{d+(m+1)+1}{t} \left(c - \frac{t}{n}\varepsilon\right)^{d+(m+1)+1-t}$$

$$\cdot \left(\varepsilon + \frac{t}{n}\right)^{t-1} \Bigg]$$

This completes the proof of Lemma 1.

Expression (2.10) is evaluated from (2.12) as follows. In (2.10) make the change of variables

$$(2.13) \qquad \begin{aligned} U_{k+2} - U_{k+1} &= v_1 \\ &\vdots \\ U_{\ell+1} - U_{k+1} &= v_{\ell-k} \end{aligned}$$

with Jacobian unity. (2.10) becomes

$$\frac{1}{(n-\ell-1)!} \int_0^{\theta} \int_{v_1}^{\frac{1}{n}-\theta} \cdots \int_{v_{\ell-k-1}}^{\frac{\ell-k-1}{n}+\theta} (1-U_{k+1}-v_{\ell-k})^{n-\ell-1} dv_{\ell-k} \cdots dv_1$$

where $\theta = \frac{k+1}{n} + \varepsilon - U_{k+1}$

By (2.11) this equals

$$\frac{1}{(n-\ell-1)!}\, I_{\ell-k-1}(1-U_{k+1}, n-\ell-1, \varepsilon) = \frac{1}{(n-k-1)!}\Bigg[(1-U_{k+1})^{n-k-1} -$$

$$\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)\sum_{t=0}^{\ell-k-1}\binom{n-k-1}{t}\left(1-\varepsilon-\frac{k+t+1}{n}\right)^{n-t-k-1}\left(\frac{k+t+1}{n}+\varepsilon-U_{k+1}\right)^{t-1}\Bigg]$$

$$= \frac{1}{(n-k-1)!}\Bigg[(1-U_{k+1})^{n-k-1} - \left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)\cdot$$

$$\sum_{\nu=k+1}^{\ell}\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu}\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)^{\nu-k-2}\Bigg].$$

Summarizing our progress to this point, we have

$$I = \int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}\frac{(1-U_{k+1})^{n-k-1}}{(n-k-1)!}\,dU_{k+1}\cdots dU_1$$

$$(2.14)\quad -\int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon}\cdots\int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0}\cdots\int_{U_k}^{U_0}\frac{\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right)}{(n-k-1)!}\sum_{\nu=k+1}^{\ell}\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu}\cdot$$

$$\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)^{\nu-k-2}dU_{k+1}\cdots dU_1 .$$

In the second integral in (2.14) write

$$\left(\frac{k+1}{n}+\varepsilon-U_{k+1}\right) \quad \text{as} \quad \left(\frac{k+1}{n}-\frac{\nu}{n}\right)+\left(\frac{\nu}{n}+\varepsilon-U_{k+1}\right)$$

and place this factor inside the summation appearing in the integrand.

Then we have

$$I = \frac{1}{(n-k-1)!}\Big\{S(1, n-k-1)$$

$$(2.15) \qquad - \sum_{\nu=k+2}^{\ell} \left(\frac{k+1-\nu}{n}\right)\binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu} S\left(\frac{\nu}{n}+\varepsilon, \nu-k-1\right)$$

$$- \sum_{\nu=k+1}^{\ell} \binom{n-k-1}{n-\nu}\left(1-\varepsilon-\frac{\nu}{n}\right)^{n-\nu} S\left(\frac{\nu}{n}+\varepsilon, \nu-k-1\right)\Big\}$$

where

$$(2.16) \qquad S(a,b) = \int_0^{\varepsilon}\int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots \int_{U_j}^{\frac{j}{n}+\varepsilon}\int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} (a-U_{k+1})^b \, dU_{k+1} \cdots dU_1 .$$

We now wish to evaluate $S(a,b)$. To this end let us first examine

$$(2.17) \qquad R_k(a,b) = \int_{U_{j+1}}^{U_0} \cdots \int_{U_k}^{U_0} (a-U_{k+1})^b \, dU_{k+1} \cdots dU_{j+2} .$$

LEMMA 2:

$$(2.18) \qquad R_k(a,b) = \frac{b!}{(b+k-j)!} \sum_{r=0}^{b} \binom{b+k-j}{r} (a-U_0)^r (U_0-U_{j+1})^{b+k-j-r}$$

PROOF: By induction on k.

1) For $k = j+1$.

$$R_{j+1}(a,b) = \int_{U_{j+1}}^{U_0} (a-U_{j+2})^b \, dU_{j+2}$$

$$= \frac{(a-U_{j+1})^{b+1}}{b+1} - \frac{(a-U_0)^{b+1}}{b+1}$$

The formula yields

$$\frac{b!}{(b+1)!}\sum_{r=0}^{b}\binom{b+1}{r}(a-U_o)^r(U_o-U_{j+1})^{b+1-r}$$

$$=\frac{1}{b+1}\left[\sum_{r=0}^{b+1}\binom{b+1}{r}(a-U_o)^r(U_o-U_{j+1})^{b+1-r}-(a-U_o)^{b+1}\right]$$

$$=\frac{1}{b+1}\left[(a-U_{j+1})^{b+1}-(a-U_o)^{b+1}\right].$$

2) Assume the result true for all k, $j+2\leq k\leq m$. Then

$$K_{m+1}(a,b)=\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}\int_{U_{m+1}}^{U_o}(a-U_{m+2})^b dU_{m+2}\cdots dU_{j+2}$$

$$=\frac{1}{b+1}\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}(a-U_{m+1})^{b+1}dU_{m+1}\cdots dU_{j+2}$$

$$-\frac{(a-U_o)^{b+1}}{b+1}\int_{U_{j+1}}^{U_o}\cdots\int_{U_m}^{U_o}dU_{m+1}\cdots dU_{j+2}$$

$$=\frac{1}{b+1}K_m(a,b+1)-\frac{(a-U_o)^{b+1}}{b+1}\cdot\frac{(U_o-U_{j+1})^{m-j}}{(m-j)!}$$

$$=\frac{1}{b+1}\cdot\frac{(b+1)!}{(b+1+m-j)!}\sum_{r=0}^{b+1}\binom{b+1+m-j}{r}(a-U_o)^r(U_o-U_{j+1})^{b+1+m-j-r}$$

$$-\frac{(a-U_o)^{b+1}}{b+1}\cdot\frac{(U_o-U_{j+1})^{m-j}}{(m-j)!}$$

$$=\frac{b!}{[b+(m+1)-j]!}\sum_{r=0}^{b}\binom{b+(m+1)-j}{r}(a-U_o)^r(U_o-U_{j+1})^{b+(m+1)-r}$$

This completes the proof of Lemma 2.

Thus we have now:

$$(2.19)\quad S(a,b) = \frac{b!}{(b+k-j)!} \sum_{r=0}^{b} \binom{b+k-j}{r} (a-U_o)^r \int_0^{\varepsilon} \int_{U_1}^{\frac{1}{n}+\varepsilon} \cdots$$

$$\cdot \int_{U_j}^{\frac{j}{n}+\varepsilon} (U_o - U_{j+1})^{b+k-j-r}\, dU_{j+1} \cdots dU_2 dU_1 .$$

The integral in the above expression is easily evaluated by Lemma 1. In the notation of the lemma, this integral is equal to

$$T_j(U_o,\ b+k-j-r) = \frac{(b+k-j-r)!}{(b+k+1-r)!} \left[U_o^{\,b+k+1-r} - \varepsilon \sum_{t=0}^{j} \binom{b+k+1-r}{t} \cdot \right.$$

$$\left. (U_o - \frac{t}{n} - \varepsilon)^{b+k+1-r-t} (\varepsilon + \frac{t}{n})^{t-1} \right] .$$

Summarizing our results, we obtain

LEMMA 3:

$$(2.20)\quad S(a,b) = \frac{b!}{(b+k+1)!} \sum_{r=0}^{b} \binom{b+k+1}{r} (a-U_o)^r \left[U_o^{\,b+k+1-r} - \varepsilon \sum_{t=0}^{j} \binom{b+k+1-r}{t} \cdot \right.$$

$$\left. (U_o - \frac{t}{n} - \varepsilon)^{b+k+1-r-t} (\varepsilon + \frac{t}{n})^{t-1} \right] .$$

where $S(a,b)$ is defined by (2.16).

Substituting into (2.15) and after a bit of not difficult algebra, we obtain the

THEOREM: For given hypothesis $H \in (F)$ and for all alternatives $G \in (F)$ such that $G(x) \leq H(x)$ for all x and such that $H(X_o) - G(X_o) = \delta$ for given δ and some X_o, the power of the test described in the introduction is at least as large as $(1-n!\,I\ )$ where

(2.21)

$$n!\, I = \sum_{r=0}^{n-k-1} \binom{n}{r} (1 - U_o)^r U_o^{\,n-r}$$

$$-\varepsilon \sum_{r=0}^{n-k-1} \sum_{t=0}^{j} \binom{n}{r,t} (1-U_o)^r (U_o - \tfrac{t}{n} - \varepsilon)^{n-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

$$+ \sum_{\nu=k+2}^{\ell} \sum_{r=0}^{\nu-k-2} \binom{n-1}{n-\nu,r} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r U_o^{\nu-1-r}$$

$$-\varepsilon \sum_{\nu=k-2}^{\ell} \sum_{r=0}^{\nu-k-2} \sum_{t=0}^{j} \binom{n-1}{n-\nu,r,t} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r \,.$$

$$(U_o - \tfrac{t}{n} - \varepsilon)^{\nu-1-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

$$- \sum_{\nu=k+1}^{\ell} \sum_{r=0}^{\nu-k-1} \binom{n}{n-\nu,r} (1 - \varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r U_o^{\nu-r}$$

$$+ \varepsilon \sum_{\nu=k+1}^{\ell} \sum_{r=0}^{\nu-k-1} \sum_{t=0}^{j} \binom{n}{n-\nu,r,t} (1-\varepsilon - \tfrac{\nu}{n})^{n-\nu} (\tfrac{\nu}{n} + \varepsilon - U_o)^r .$$

$$(U_o - \tfrac{t}{n} - \varepsilon)^{\nu-r-t} (\varepsilon + \tfrac{t}{n})^{t-1}$$

and where

$$\binom{n}{a,b,\ldots} \quad \frac{n!}{a!b!\ldots(n-a-b-\ldots)!}$$

This lower bound as a function of $H(X_o)$, δ, ε cannot be improved, since for any given $H \in (F)$, X_o, ε, and δ we can construct a $G \in (F)$ arbitrarily close to G^*.

The upper bound for the power is the same as that obtained in [3] although a different alternative is considered there. The upper bound is

$$\sum_{i=0}^{m} \binom{n}{i} \left(1-\varepsilon+\delta-\frac{i}{n}\right)^{n-i}\left(\varepsilon-\delta+\frac{i}{n}\right)^{i-1} \quad \text{for } \varepsilon \geq \delta$$

where $m = [n(1-\varepsilon+\delta)]$ and the upper bound 1 for $\varepsilon < \delta$. This upper bound cannot be improved.

REFERENCES

(1) A. Wald and J. Wolfowitz, "Confidence limits for continuous distribution functions", Ann. Math. Stat. vol. 10 (1939), pp. 105-118.

(2) Z. W. Birnbaum and Fred H. Tingey, "One-sided confidence contours for probability distribution functions", Ann. Math. Stat. vol. 22 (1951), pp. 592-596

(3) Z. W. Birnbaum, "On the power of a one-sided test of fit for continuous probability functions", Ann. Math. Stat. vol. 24 (1953), pp. 484-489

Chief of Naval Research
Office of Naval Research
Washington 25, D. C.
Attn: Code 432
(Mathematics Branch) 5

Dr. E. Lukacs
Head, Statistics Branch
Office of Naval Research
Department of the Navy
Washington 25, D. C. 5

Scientific Section
Office of Naval Research
Department of the Navy
1000 Geary Street
San Francisco 9, Calif. 2
Attn: Dr. J. Wilkes 1

Director, Naval Research
Laboratory
Washington 25, D. C.
Attn: Technical Information
Officer 6

Office of the Assistant Naval
Attache for Research
Naval Attache
American Embassy
Navy No. 100
Fleet Post Office
New York, New York 2

Director
Office of Naval Research
Branch Office
844 North Rush Street
Chicago 11, Illinois 1

Director
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York 1

Director
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California 1

Commander
U.S. Naval Ordnance Test Station
Inyokern, China Lake, California 1

Chief of Naval Operations
Operation Evaluation Group-OP 374
The Pentagon
Washington 25, D. C. 1

Office of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438
(Mechanics Branch) 2

Office of Naval Research
Logistics Branch - Code 436
T-3 Building
Washington 25, D. C. 1

ASTIA Documents Service Center
Knott Building
Dayton 2, Ohio 5

Headquarter, USAF
Director of Research and
Development
Washington 25, D. C. 1

Assistant Chief of Staff, G-4
for Research and Development
U. S. Army
Washington 25, D. C. 1

Chairman
Research and Development Board
The Pentagon
Washington 25, D. C. 1

Office of Technical Services
Department of Commerce
Washington 25, D. C. 1

Commander, Air Force Personnel
and Training Research Center
Attn: Director, Personnel Research Laboratory
Lackland Air Force Base, Texas 1

National Bureau of Standards
Institute for Numerical Analysis
405 Hilgard Avenue
Los Angeles 24, California 2

Chief, Statistical Engineering
Laboratory
National Bureau of Standards
Washington 25, D. C. 1

RAND Corporation
1500 Fourth Street
Santa Monica, California 1

Applied Mathematics and
Statistics Laboratory
Stanford University
Stanford, California 3

Professor Carl B. Allendoerfer
Department of Mathematics
University of Washington
Seattle 5, Washington 1

Professor W. G. Cochran
Department of Biostatistics
The Johns Hopkins University
Baltimore S., Maryland 1

Professor Benjamin Epstein
Department of Mathematics
Wayne University
Detroit 1, Michigan 1

Professor Herbert Solomon
Teachers College
Columbia University
New York, New York 1

Professor W. Allen Wallis
Committee on Statistics
University of Chicago
Chicago 37, Illinois 1

Professor J. Wolfowitz
Department of Mathematics
Cornell University
Ithaca, New York 1

Department of Mathematical Statistics
University of North Carolina
Chapel Hill, North Carolina 2

Professor J. Neyman
Statistical Laboratory
University of California
Berkeley, California 2

Professor S. S. Wilks
Department of Mathematics
Princeton, New Jersey 1