ON THE POWER OF A ONE-SIDED TEST OF FIT AGAINST STOCHASTICALLY COMPATIBLE ALTERNATIVES

By

Z. W. Birnbaum and Ernest M. Scheuer

University of Washington

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Laboratory of Statistical Research
Department of Mathematics
University of Washington
Seattle, Washington
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I INTRODUCTION

Let $F$ be the continuous cumulative distribution function (c.d.f.) for the random variable (r.v.) $X$ and $F$, the empirical c.d.f. determined by the ordered sample $X_1, X_2, \ldots, X_n$ of $X$. $F_n$ is defined by

\[
F_n(x) = \begin{cases} 
0 & \text{for } x < X_1 \\
\frac{k}{n} & \text{for } X_k \leq x < X_{k+1}; \ k = 1, \ldots, n-1 \\
1 & \text{for } x \geq X_n
\end{cases}
\]

(1.1)

It is known [1] that the probability $P \{ F(x) \leq F_n(x) + \varepsilon, \text{all } x \}$ is a function independent of $F$. We will use this function to test the hypothesis $F = H$ against the alternative $F = G$. The power of the test will be studied for alternatives $G$ such that $G(x) \leq H(x)$ for all $x$ and such that

\[
\sup_{-\infty < x < -\infty} [H(x) - G(x)] = \delta, \text{ with pre-assigned } \delta > 0.
\]

Alternatives of this kind will be called "stochastically comparable, at distance $\delta$ from $H."$ For brevity's sake we shall refer to them as alternatives (A).

We assume throughout that $H \in (F)$, $G \in (F)$ where $(F)$ is the set of all continuous strictly increasing c.d.f.'s.

We test $F = H$ against $F = G$ by the following procedure. To have a test of size $\alpha$ for sample size $n$ we will use the value $\varepsilon_{n, \alpha}$ from Table 1 in [2], obtain an ordered sample $X_1, X_2, \ldots, X_n$ of $X$, determine the empirical c.d.f. of $X$ and reject $H$ if and only if the inequality

(1.2) \[ H(x) \leq F_n(x) + \varepsilon_n, \alpha \]

fails to hold for all real $x$. 
The power of this test is the complementary probability to

\[ \text{Pr} = \frac{1}{n^2} \sum_{i=1}^{n} P(H(x_i) \leq \frac{i-1}{n} + \varepsilon) \text{ for all } x \in \mathbb{C}. \]

Since (1.2) is satisfied for all \( x \) if and only if

\[ H(x_i) \leq \frac{i-1}{n} + \varepsilon \text{ for } i = 1, \ldots, n \]

we have (writing hereafter \( \varepsilon \) for \( \varepsilon_n \)) and noting that \( H(x) \leq 1 \) for all \( x \)

\[ \text{Pr} = P \left( \frac{1}{n} \sum_{i=1}^{n} H(x_i) \leq \varepsilon, i = 1, \ldots, n \right| 0 \}

Define the function \( L \) by

\[ L(\varepsilon) = \begin{cases} \lim_{\varepsilon \to 0^+} (\varepsilon) & \text{for } \varepsilon \leq 0 \\ \lim_{\varepsilon \to 1^-} (\varepsilon) & \text{for } \varepsilon > 0 \end{cases} \]

Recall that \( U = G(x) \) has the rectangular distribution \( k \) in the unit interval when \( x \) has c.d.f. \( G \). We conclude

\[ \text{Pr} = P \left( U_i \leq \frac{i-1}{n} + \varepsilon, i = 1, \ldots, n \right| 0 \}

Clearly \( U_1, U_2, \ldots, U_n \) is an ordered sample of the \( U \)'s. Since the joint probability distribution of \( (U_1, U_2, \ldots, U_n) \) is equal to \( k \) for \( 0 \leq U_1 \leq U_2 \leq \cdots \leq U_n \leq 1 \) and zero elsewhere, we have

\[ L(\varepsilon) = L \left( \frac{k_{i-1}}{n} + \varepsilon, i = 1, \ldots, n \right| 0 \}

\[ \text{Pr} = \prod_{i=1}^{n} \left( \frac{1}{k_{i-1}} + \varepsilon \right) \]
If $H(x) = G(y)$, then $L(V) = V$ for $0 < V < 1$ and (1.7) reduces to

formula (3.3) in [1]. If $h$ and $d$ are given, it may be possible to evaluate $P$, and hence the lower $1-P$, from (1.7) by quadrature, or one

may compute it by numerical integration. In the special case for,

alternatives (A), it is possible to derive from (1.7) inequalities for

the error, as will be seen.
II ALTERNATIVES (a)

For given hypothesis $H$, we consider here alternatives $G$ such that $G(x) \leq H(x)$ for all $x$ and such that $\sup_{x < x_0} [H(x) - G(x)] = \delta$ for pre-assigned $\delta > 0$. In view of the assumption that $H \in (F), G \in (F)$, the supremum is actually attained, say at $x_0$. That is,

$$H(x_0) - G(x_0) = \delta.$$

For intuitive reasons one may expect that under these restrictions the power of our test will be close to its minimum when $G$ is close to the function $G^*$ defined by

$$(2.1) \quad G^*(x) = \begin{cases} \underline{H}(x) & \text{for } x < \underline{H}^{-1}(U_0) \\ U_0 & \text{for } \underline{H}^{-1}(U_0) \leq x < x_0 \\ \overline{H}(x) & \text{for } x \geq x_0 \end{cases}$$

where $U_0 = H(x_0) - \delta = G(x_0)$.

To verify this conjecture we consider

$$(2.2) \quad G^*(V) = \begin{cases} V & \text{for } 0 \leq V < U_0 \\ U_0 & \text{for } U_0 \leq V < V_0 \\ V_0 & \text{for } V_0 \leq V < 1 \\ 1 & \text{for } V \geq 1 \end{cases}$$

where $V_0 = H(x_0)$. 
Let
\[ j = \left\lfloor n(V_0 - \varepsilon) \right\rfloor \]
\[ k = \left\lfloor n(V_0 - \xi) \right\rfloor \]
\[ L = \left\lfloor n(l- \xi) \right\rfloor \]
where \( \left\lfloor A \right\rfloor \) is the greatest integer less than \( A \).

We want to show \( L(V) \leq L^*(V) \) for all \( V \). We note first that for \( 0 < V < 1 \)
\[ (2.4) \quad L(V) \leq V. \]

Suppose \( x = H^{-1}(V) \). Then \( V \equiv H(x) \) and \( x = H^{-1}(V) \Rightarrow L(V) \). By assumption \( G(x) \leq H(x) \), therefore \( L(V) \leq V \).

We note next that for \( j + 1 \leq \xi < k \) we have \( \frac{2}{n} + \xi \leq \frac{2}{n} + \varepsilon \leq V_0 \). Therefore
\[ (2.5) \quad L\left( \frac{2}{n} + \xi \right) \leq L(V_0) \geq U_0 \quad \text{for} \quad j + 1 \leq \xi < k. \]

Collecting up
\[ L(V) \leq V = L^*(V) \quad \text{for} \quad 0 < V < U_0 \quad \text{and} \quad V_0 < V < 1, \]
which means
\[ (2.6) \quad L\left( \frac{2}{n} + \xi \right) \leq L^*(\frac{2}{n} + \xi) \quad \text{for} \quad 0 \leq \xi \leq j \quad \text{and} \quad k + 1 \leq \xi \leq \ell. \]

Further, \( L(V) \leq 1 \) always, so
\[ (2.7) \quad L\left( \frac{2}{n} + \varepsilon \right) \leq L^*(\frac{2}{n} + \varepsilon) \quad \text{for} \quad \ell + 1 < \xi. \]

Formulas \((2.5), (2.6), (2.7)\) show us that replacing, in \((1.7)\) the function \( L \) by the function \( L^* \) in the upper limits of integration will
not decrease these limits. Hence

\[
\frac{1}{n} \int_{U_1}^{U_k} \ldots \int_{U_{k+1}}^{U_{k+2}} \ldots \int_{U_{k+l}}^{U_{k+l+1}} \ldots \int_{U_{n-1}}^{U_n} \frac{dU_1 \ldots dU_{k+1} \ldots dU_{k+l} \ldots dU_{l+1} \ldots dU_{n-1}}{U_1 \ldots U_{k+1} \ldots U_{k+l} \ldots U_{l+1} \ldots U_{n-1}}.
\]

Denote the integral on the right by \(I\). We proceed to evaluate \(I\).

By (2.1) of 2, we have

\[
\int_{U_{k+1}}^{U_{k+l+1}} \ldots \int_{U_{n-1}}^{U_n} \frac{dU_1 \ldots dU_{k+1} \ldots dU_{k+l} \ldots dU_{l+1} \ldots dU_{n-1}}{U_1 \ldots U_{k+1} \ldots U_{k+l} \ldots U_{l+1} \ldots U_{n-1}} = \frac{(1-U_{k+l+1})^{n-k-l}}{(n-k-l)l},
\]

so

\[
I = \int_{U_1}^{U_k} \ldots \int_{U_{k+i}}^{U_{k+i+1}} \ldots \int_{U_{n-1}}^{U_n} \frac{dU_1 \ldots dU_{k+1} \ldots dU_{k+l} \ldots dU_{l+1} \ldots dU_{n-1}}{U_1 \ldots U_{k+1} \ldots U_{k+l} \ldots U_{l+1} \ldots U_{n-1}}.
\]

Next we want to evaluate

\[
\int_{U_{k+1}}^{U_{k+l+1}} \ldots \int_{U_{n-1}}^{U_n} \frac{(1-U_{k+l+1})^{n-k-l}}{(n-k-l)l} dU_{k+1} \ldots dU_{k+l+1}.
\]

Instead we evaluate a slightly more general expression. We shall have occasion to use it later in our discussion as well as now.

Let

\[
T_j(c,a,\xi) = \int_{U_1}^{U_k} \ldots \int_{U_{k+i}}^{U_{k+i+1}} \ldots \int_{U_{n-1}}^{U_n} (c-U_{j+1})^d dU_{j+1} \ldots dU_{n-1}
\]

for some positive integer \(d\).
Then we have

$$T_j(x, \epsilon) = \frac{\epsilon^{j+1}}{(a+j)!} \left[ a_{j+1} (x_a + b_{j+1} \epsilon + c_{j+1} \epsilon^2 + \cdots + d_{j+1} \epsilon^{j+1} + e_{j+1} \epsilon^{j+2} + \cdots + f_{j+1} \epsilon^{j+3} + g_{j+1} \epsilon^{j+4} + \cdots) \right]$$

**Proof:** By induction on $j$.

1. For $j = 0$, trivial.

2. Assume the result true for all $j$, let $j > 0$. Then

$$T_{j+1}(c, \epsilon) = \frac{1}{(a+j+1)!} \left[ a_{j+1} (c_{j+1} + \epsilon + c_{j+2} \epsilon^2 + \cdots + d_{j+2} \epsilon^{j+2} + e_{j+2} \epsilon^{j+3} + \cdots) \right]$$

$$= \frac{\epsilon}{(a+j)!} \left[ \cdots \right]$$

$$= \frac{\epsilon^{j+1}}{(a+j)!} \left[ \cdots \right]$$

$$(\text{See (2.2) or [2])}$$
\[ T_{m+1}(c,d,\xi) = \frac{1}{d+1} \cdot \frac{(d+1)!}{((d+1)+m+1)!} \left[ \frac{c}{d+1} \right]^{(d+1)+m+1} \]

\[ \xi \sum_{t=0}^{\infty} \binom{(d+1)+m+1}{t} (c - \frac{t}{n} - \xi) (d+1)+m+1-t \left( \xi + \frac{t}{n} \right)^{t-1} \]

\[ = \xi \sum_{t=0}^{\infty} \binom{(d+1)+m+1}{t} \left[ (c - \frac{t}{n} - \xi) \cdot \left( \frac{d+1}{(d+1)+m+1+t} \right) (d+1)+m+1-t \left( \xi + \frac{t}{n} \right)^{t-1} \right] \]

This completes the proof of Lemma 1.

Expression (2.10) is evaluated from (2.12) as follows. In (2.10) make the change of variables

\[ u_k = c - u_k \xi \]

(2.13)

\[ \vdots \]

\[ u_{q+1} = c - u_{q+1} \xi \]

with Jacobian unity. (2.10) becomes

\[ \frac{1}{(n-k-1)!} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{\xi^{k+1}}{v_1 \cdots v_{k-1}} \frac{\xi^{k+1}}{v_{k+1} \cdots v_n} \]

where \( \xi = \frac{k+1}{n} + \xi \cdot \frac{n}{k+1} \).
By (2.11) this a-crcle

\[
\frac{1}{(n-k-1)!} \left( n \cdot \frac{k}{n} - \phi \right) \frac{1}{(n-k-1)!} \left( \lambda - \frac{k}{n} \right)^{n-k-1, n} \times \left( \lambda - \frac{k}{n} \right)^{n-k-1, n}
\]

\[
\left( \frac{1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right) \sum_{t=0}^{\infty} (n-k-1) (1-\frac{\varepsilon - U_{k+1}}{n}) \left( \frac{t+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right)^{t-2}
\]

\[
= \frac{1}{(n-k-1)!} \left[ \left( 1 - \frac{k}{n} \right)^{n-k-1, n} + \frac{\varepsilon - U_{k+1}}{n} \right] \cdot
\]

\[
\sum_{t=0}^{\infty} (n-k-1) (1-\frac{\varepsilon - U_{k+1}}{n}) \left( \frac{t+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right)^{t-2}
\]

Summarizing our progress to this point, we have

\[
I = \int_0^1 \int_0^{U_k} \cdots \int_0^{U_{k+1}} \cdots \int_0^{U_{k+1}} \left( \frac{t+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right) \frac{(1-k+1)}{(n-k-1)!} \, du_{k+1} \cdots du_1
\]

\[
(2.14) \quad \left( \frac{1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right)^{n-k-1} \sum_{t=0}^{\infty} \left( \frac{t+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right) \frac{(1-k+1)}{(n-k-1)!} \left( \frac{t+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right)^{t-2}
\]

In the second integral in (2.14) write

\[
\left( \frac{k+1}{n} + \frac{\varepsilon - U_{k+1}}{n} \right) \text{ as } \left( \frac{k+1}{n} - \frac{1}{n} \right) + \left( \frac{\varepsilon - U_{k+1}}{n} \right)^{n-k-1}
\]

and place this factor inside the summation appearing in the integrand.

Then we have
\[ I = \frac{1}{(n-k-1)!} \sum_{r=0}^{k} \left( \frac{k!}{n-r!} \right) \left( 1 - \frac{k}{n} \right) (n-r+k) ^{n-1} (1 - r) \]

\[ = \sum_{r=k}^{n-k-1} \left( \frac{n-k-1}{n-r} \right) (n-r+k) ^{n-2} \left( \frac{k!}{(n-r)!} \right) \]

\[ \text{where} \]

\[ \sum_{r} \left( \frac{n-k-1}{n-r} \right) (n-r+k) ^{n-2} \left( \frac{k!}{(n-r)!} \right) \]

We now wish to evaluate \( S(a, b) \). To this end, let us first examine

\[ S(a, b) = \sum_{r} \left( \frac{n-k-1}{n-r} \right) (n-r+k) ^{n-2} \left( \frac{k!}{(n-r)!} \right) \]

** Lemma 2:**

\[ R_{k+1} (a, b) = \frac{b!}{(b+k-j)!} \sum_{r=0}^{b} \left( \frac{b+k-j}{r} \right) (a-U_{j+r}) ^{b} (U_{j+r} - U_{j+r+1}) ^{b+k-j} \]

** Proof:** by induction on \( k \).

1) For \( k = j + 1 \).
The formula yields

\[
\frac{b!}{(b+1)!} \sum_{r=0}^{b} \frac{(b+1)}{r}(a-U_0)^r(U_0-U_{j+1})^b + 1 - r
\]

\[
= \frac{1}{b+1} \left[ \sum_{r=0}^{b+1} \frac{b+1}{r}(a-U_0)^r(U_0-U_{j+1})^b + 1 - r - (a-U_0)^{b+1} \right]
\]

\[
= \frac{1}{b+1} \left[ (a-U_{j+1})^{b+1} - (a-U_0)^{b+1} \right].
\]

2) Assume the result true for all \( k, j + 2 < k \leq m \). Then

\[
\frac{1}{b+1} \int_{U_{j+1}}^{U_m} \cdots \int_{U_{j+2}}^{U_m} (a-U_{j+1})^{b+1} \cdots du_{j+2}
\]

\[
= \frac{1}{b+1} \int_{U_{j+1}}^{U_m} \cdots \int_{U_{j+2}}^{U_m} (a-U_{j+1})^{b+1} \cdots du_{j+2}
\]

\[
= \frac{1}{b+1} \int_{U_{j+1}}^{U_m} \cdots \int_{U_{j+2}}^{U_m} (a-U_{j+1})^{b+1} \cdots du_{j+2}
\]

\[
= \frac{1}{b+1} \int_{U_{j+1}}^{U_m} \cdots \int_{U_{j+2}}^{U_m} (a-U_{j+1})^{b+1} \cdots du_{j+2}
\]

\[
= \frac{1}{b+1} \int_{U_{j+1}}^{U_m} \cdots \int_{U_{j+2}}^{U_m} (a-U_{j+1})^{b+1} \cdots du_{j+2}
\]

This completes the proof of Lemma 2.
Thus we have now:

\[(2.19) \quad S(a,b) = \frac{b!}{(b+k-j)!} \sum_{r=0}^{b} \binom{b+k-j}{r} (a-U_0)^r \int_0^\infty \int_0^1 \int_0^1 \cdots \int_0^1 \cdots \int_0^1 \cdot \frac{1}{U_j} (U_j-U_{j+1})^{b+k-j-r} dU_j dU_{j+1} \cdots dU_2 dU_1.\]

The integral in the above expression is easily evaluated by Lemma 1. In the notation of the lemma, this integral is equal to

\[T_j(U_j, b+k-j-r) = \frac{(b+k-j-r)!}{(b+k-j+r)!} \int_0^{b+k-j-r} (U_j - \varepsilon)^{b+k-j-r} (\varepsilon + \frac{t}{n})^{t-1} \cdots \left(U_0 - \frac{t}{n} - \varepsilon\right)^{b+k-j-r-t} (\varepsilon + \frac{t}{n})^{t-1}\]

Summarizing our results, we obtain

**Lemma 3:**

\[(2.20) \quad S(a,b) = \frac{b!}{(b+k+j)!} \sum_{r=0}^{b} \binom{b+k+j}{r} (a-U_0)^r \int_0^\infty \int_0^1 \int_0^1 \cdots \int_0^1 \cdots \int_0^1 \cdot \frac{1}{U_j} (U_j-U_{j+1})^{b+k+j-r} dU_j dU_{j+1} \cdots dU_2 dU_1.\]

where \(S(a,b)\) is defined by (2.16).

Substituting into (2.15) and after a bit of not difficult algebra, we obtain the

**Theorem:** For given hypothesis \(H \in \mathcal{F}\) and for all alternatives \(G \in \mathcal{F}\) such that \(G(x) \leq H(x)\) for all \(x\) and such that \(H(x_0) - G(x_0) = \delta\) for given \(\delta\) and some \(x_0\), the power of the test described in the introduction is at least as large as \((1-n11)\) where
\[(2.21)\]

\[
\begin{align*}
\nu_l(I) &= \sum_{r=0}^{n-k-1} \binom{n}{r} (1 - U_0)^r U_0^{n-r} \\
&\quad - \varepsilon \sum_{r=0}^{n-k-1} \sum_{t=0}^{\nu_{r-1}} (1 - U_0)^r (U_0 - \frac{t}{n} - \varepsilon)^{n-r-t} (\varepsilon + \frac{t}{n})^{t-1} \\
&\quad + \sum_{\nu=k+1}^{l} \sum_{r=0}^{\nu-1} (1 - U_0)^r (\varepsilon - \frac{t}{n})^{n-\nu} (U_0 - \varepsilon)^{U_0^{\nu-1}} (U_0 - \varepsilon - U_0)^r \\
&\quad - \varepsilon \sum_{\nu=k+1}^{l} \sum_{r=0}^{\nu-1} (1 - U_0)^r (\varepsilon - \frac{t}{n})^{n-\nu} (U_0 - \varepsilon - U_0)^r U_0^{U_0^{\nu-1}} \\
&\quad + \varepsilon \sum_{\nu=k+1}^{l} \sum_{r=0}^{\nu-1} (1 - U_0)^r (\varepsilon - \frac{t}{n})^{n-\nu} (U_0 - \varepsilon - U_0)^r (U_0 - \varepsilon)^{U_0^{\nu-1}} (U_0 - \varepsilon - U_0)^r (U_0 - \varepsilon - U_0)^r \\
&\quad = \sum_{t=0}^{n-1} n^t (a, b, \ldots) n_i (n - a - b - \ldots)!
\end{align*}
\]

This lower bound as a function of $\mathcal{H}(X_0)$, $\delta$, $\varepsilon$ cannot be improved, since for any given $\mathcal{H}(F)$, $X_0$, $\varepsilon$, and $\delta$ we can construct a $G \in \mathcal{G}(F)$ arbitrarily close to $u^*$. 
The upper bound for the power is the same as that obtained in [3] although a different alternative is considered there. The upper bound is

\[ \sum_{i=0}^{m-1} \binom{m}{i} \left(1-\varepsilon+\delta_{i} \right)^{m-1} (\varepsilon-\delta_{i})^{i-1} \text{ for } \varepsilon \geq \delta \]

where \( m = \lfloor n(1 - \varepsilon + \delta) \rfloor \) and the upper bound 1 for \( \varepsilon < \delta \). This upper bound cannot be improved.
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National Bureau of Standards
Institute for Numerical Analysis
400 Hilgard Avenue
Los Angeles 24, California

Chief, Statistical Engineering Laboratory
National Bureau of Standards
Washington 25, D. C.

RAND Corporation
1500 Fourth Street
Santa Monica, California

Applied Mathematics and Statistics Laboratory
Stanford University
Stanford, California

Professor Carl B. Allendoerfer
Department of Mathematics
University of Washington
Seattle 5, Washington

Professor W. G. Cochran
Department of Biostatistics
The Johns Hopkins University
Baltimore 5, Maryland

Professor Benjamin Epstein
Department of Mathematics
Wayne University
Detroit 1, Michigan

Professor Herbert Solomon
Teachers College
Columbia University
New York, New York

Professor W. Allen Wallis
Committee on Statistics
University of Chicago
Chicago 37, Illinois

Professor J. Wolfowitz
Department of Mathematics
Cornell University
Ithaca, New York

Department of Mathematical Statistics
University of North Carolina
Chapel Hill, North Carolina

Professor J. Neyman
Statistical Laboratory
University of California
Berkeley, California

Professor S. S. Wilks
Department of Mathematics
Princeton, New Jersey