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THE EFFECT OF SHEAR STRESSES
ON THE CARRYING CAPACITY OF I-BEAMS

by

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GRADUATE DIVISION OF APPLIED MATHEMATICS

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The Effect of Shear Stresses on the Carrying Capacity of I-Beams

by

Carl-Fredrik A. Leth

Abstract.

This paper considers the bending of a cantilever I-beam of a ductile metal, such as mild steel, that is loaded by a transverse force at the free end. Taking the length of the beam into account the load carrying capacity can be expressed in terms of the moment at the built-in end that is in equilibrium with the maximum transverse force at which collapse of the beam is imminent, owing to the development of regions of plastic flow. The influence of the length of the beam upon this limit moment is studied in the paper.

As the simplest approximation of the limit moment the fully plastic moment can be taken. From experimental evidence it is known that sufficiently long beams can support a force that is in equilibrium with the fully plastic moment. But it is also observed that for beams that have both a short length and an I-shaped cross-section, the limit moment may be considerably less than the fully plastic moment. The aim of the paper is to obtain theoretical estimates of the maximum moment that these I-beams can develop at the built-in end, assuming that the material exhibits ideal plasticity.

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At present a complete three-dimensional solution of the problem seems to be out of the question. Even for a two-dimensional model of the problem it would be very difficult to obtain a complete elastic-plastic solution. The limit analysis theorems of Drucker, Prager and Greenberg are therefore applied. These furnish upper and lower bounds for the limit moment. Two approximate lower bounds are obtained by constructing two different admissible stress fields for a two-dimensional model of the problem. Only the second bound gives estimates of the limit moment for short beams, but both bounds are applicable to long beams. For short beams an upper bound is obtained from a kinematically admissible velocity field, using the appropriate limit analysis theorem.

The first stress field is based on a natural extension of the conventional elastic theory of the bending of beams, modifying this in such a way that at loads above the load at which plastic flow first occurs the yield condition is not violated anywhere. The second stress field is artificial in the sense that it is unrelated to the stress distribution of an elastic beam. Nevertheless, it furnishes in a much shorter way than the first a lower bound for the limit moment, valid for both short and long beams.

Both stress fields yield values of the limit moment which tend towards the fully plastic moment for long beams as the ratio of the beam length to depth is increased. For short beams the second stress field gives a value of the limit moment which differs considerably from the fully plastic moment. For this
reason a velocity field is constructed so that an upper bound is obtained for short beams. This velocity field is based on the failure of the short I-beam primarily because of shear in the web. It is shown by examples that the upper and lower bounds are very close for short beams.

The theoretical results obtained in this paper are compared with the reported experimental data that are concerned with the influence of shear stresses. The experiments give a reasonable confirmation of the theoretical results both for the value of the limit moment and for the computed critical length of the beam below which the carrying capacity rapidly decreases.

A. Introduction.

The problem of the elastic-plastic bending of beams is so complex that at present no rigorous complete solutions have been found. In this paper is presented an approximate treatment of a cantilever I-beam which carries a transverse force parallel to the web at the "free" end, Fig. 1. The purpose is to study the effects of the shear stresses on the carrying capacity of the beam. These may gain importance because of the difference in width of the flange and the web.

In the elastic analysis of the general problem of a beam under end loading an exact solution can be obtained by the sem-inverse method of Saint-Venant. For commonly used beam sections, however, this solution is too complicated to be used in normal engineering practice. Instead an approximate solution is used that is based upon an assumption, introduced by J. Bernoulli, concerning the deformation of the beam. The deflection
of the beam is assumed to take place in such a way that points which initially lie in a plane normal to the center-line after the deformation also form a plane that is normal to the deflected center-line. This condition together with the equilibrium equations and Hooke's law yields results that do not fulfill the compatibility equations. Furthermore if the cross-section is not rectangular the condition of a stress-free cylindrical surface of the beam is violated. These inconsistencies are generally of minor importance, however, as many experiments have shown.

In considering the bending of the beam of Fig. 1 beyond the elastic limit, we have formulated a problem that is of fundamental importance in the theory of limit analysis of structures. It is not appropriate in this paper to discuss the procedures of limit analysis. Such discussions can be found e.g. in the book by Van den Broek [1]* and the papers by Baker [2] and Symonds and Neal [3]. It is sufficient to state that limit analysis of beams and frames is an analysis of failure of these structures associated with the formation of "plastic hinges" in the members of the structure. Plastic hinge sections are defined as those across which rotations of arbitrary magnitudes can occur while a constant bending moment, called the limit moment, \( M_o \), is transmitted. In this paper we discuss the computation of the hinge moment at the base of a cantilever I-beam, since this can be regarded as the basic structural element

* Numbers in square brackets refer to the Bibliography at the end of the paper.
in any continuous beam or rigid frame. We shall compare the limit moment for such a cantilever with the approximation to the limit moment that is commonly used in the applications of limit analysis, namely the "fully plastic moment", $M_p$. This latter moment is now considered in detail.

Consider first a beam subjected to pure bending. Let the beam have two planes of symmetry and let the moment act in one of them, Fig. 2. Assume the material to follow the perfectly elastic-plastic stress-strain law, i.e., the stress-strain diagram in simple tension of Fig. 3. Moreover let it be assumed that each fiber is in a state of simple tension or compression, so that $\sigma_x$ is the only non-vanishing stress. The condition that cross-sections remain plane in the elastic part of the beam then yields the distribution of stress over the cross-section that is depicted in Fig. 4. In Fig. 4a the outmost fiber has not reached the yield stress, but in 4b the moment has increased so that the outer part of the cross-section becomes plastic, while the inner part remains in the elastic state. In Fig. 4c the curvature of the beam has taken such a large value that the elastic core in the center can be neglected and the plastic regions are considered to cover the entire cross-section. The bending moment corresponding to this last state is called the fully plastic moment.

In general for a beam having two planes of symmetry, in one of which the load is applied, the fully plastic moment is given by

$$M_p = YZ_p$$
where $Z_p$ is twice the moment of the cross-section area on one side of the middle surface, with respect to the transverse axis of symmetry. In the special case of an I-beam, the fully plastic moment can be considered to consist of two parts, namely the contributions from the flanges and from the web. We then have

$$M_p = M_{pf} + M_{pw}$$

where $M_{pf} = b_1(a^2 - c^2)Y$ and $M_{pw} = b c^2 Y$ if the web and flange are assumed to be rectangles, Fig. 5.

Note that the stress distributions of Fig. 4 are not an exact solution even for a rectangular beam in pure bending, as was pointed out by Hill [4]; these stresses are not consistent with continuity of displacements at the elastic-plastic boundary.

In the more general problem of the bending of the beam in the presence of shear forces, the moment varies along the beam, and it is necessary to remove the previous assumption that $\tau_{xz}$ is zero. It is shown in the book of Prager and Hodge [5], see also Hodge [6], that $\tau_{xz}$ must vanish in the plastic region that spreads in from the outer fibers. Hence the distribution of the longitudinal stress, depicted in Fig. 4, remains unchanged if the other assumptions of the analysis for simple bending are retained. The depth of the plastic region then varies according to the distribution of the moment along the beam. Figure 6 shows the plastic region in the cantilever beam loaded with a shear force at the free end. As the elastic region must carry the total shear force, the shear stresses increase in magnitude towards the built-in end at the center of the beam. Failure by
collapse occurs when the limit moment in the presence of shear is reached at the built-in sections; deflections would then continue under constant load, if geometry changes are neglected, with rotations of arbitrary amounts at the plastic hinge at the built-in end. In most applications of limit analysis the component beams are assumed to be sufficiently long with respect to their depth so that the shear stresses can be neglected in computing the limit moment. Thus the fully plastic moment is taken in these cases as a good approximation of the actual limit moment.

An early paper dealing with the shear stresses at the section of failure of the beam was written by Stüssi [7]. He considers a beam under the influence of a moment and a shear force. Assuming that the longitudinal stress follows a general stress-strain law in simple tension for mild steel and considering plane cross-sections to remain plane over the entire cross-section, the distribution of shear stress is obtained with the help of the equilibrium equation. The fact that the value of the longitudinal stress at yield is affected by the presence of these shear stresses is neglected. The shear stress at the center of the critical cross-section at a specified stage of the loading is considered to form the condition for the failure of beam.

A recent investigation of the shear stresses at the most highly stressed section has been presented by Horne [8]. A rectangular or an I-shaped cross-section under the assumption of plane stress and the Tresca yield condition is treated. At
failure two different plastic regions appear. One of these contains the outer fibers of the beam at the built-in end where longitudinal stresses are large; the other plastic region develops at the center of the beam where shear stresses predominate. The assumption that plane cross-sections remain plane during the deformation is adopted in the elastic region. In the plastic regions account is taken of the equilibrium and yield conditions, but not of strains. It is not possible in this solution to satisfy all the matching conditions of the stresses at the plastic-elastic boundaries. For the case of I-beams the possibility of a violation of the yield condition at the flange-web junction is not investigated.

A different viewpoint is presented in a paper by Onat and Shield [9]. The same problem as in Fig. 1 is treated for a rectangular cross-section with the assumptions of plastic-rigid material and plane strain conditions. The exact solution (fulfilling both stress- and flow-conditions) for the region near the built-in end is obtained.

In the experimental studies of the bending of beams beyond the elastic limit, there are many results showing that for long beams the fully plastic moment provides a good approximation to the limit moment. A survey of these studies has been presented by Roderick and Phillips [10]. On the other hand, relatively few experiments have been made whose aim was to study the behavior of the beam when shear effects are important. Apparently the only extensive tests of this nature are those made by Baker and Roderick [11], Hendry [12], and Johnston,
Yang and Beedle [13]. The comparison of the results of this paper with these experiments is postponed until after the analysis has been described.

B. General concepts.

The analysis in this paper is based upon the two limit analysis theorems of Drucker, Prager and Greenberg [14]. The main part of the paper is concerned with the first of these theorems, and we begin by defining the terms and concepts that are used in it.

A yield function $\phi$ is a positive definite function of the stress components only, such that plastic flow can occur when $\phi = 1$. In a stress-free state we have $\phi = 0$ and the elastic range consists of $0 < \phi < 1$. We then define a safe state of stress as a state for which $\phi < 1$ throughout the structure. On the other hand collapse is defined as the state for which plastic flow would occur under constant load if the accompanying changes in the geometry of the structure were disregarded. Moreover we call a stress system statically admissible when it satisfies the equilibrium equations and is consistent with the surface tractions on the boundary of the body. The first limit analysis theorem then states that if a safe statically admissible state of stress can be found at each stage of loading, collapse will not occur under the given loading schedule. Thus a load at which a safe statically admissible stress field can be constructed is a lower bound for the collapse load of a structure.

In the main part of this paper we apply the theorem stated above to the problem of estimating a lower bound for the
maximum end force that can be applied to a cantilever beam, Fig. 1. As discussed later in this section we find it necessary to make certain approximations in the construction of safe statically admissible stress fields for this problem. Since the conditions of the theorem are not satisfied exactly, the maximum loads obtained in our analysis cannot be regarded as true lower bounds of the collapse load of the problem. However, it is believed that the nature of the approximations is such that the results will be of practical usefulness, and will serve as a guide to further experiments.

As it is part of our purpose to study plastic regions in the beam we consider stress systems that fulfill the yield condition $\Phi \leq 1$, the equilibrium equations, and agree with the specified surface tractions. We then know that the corresponding load either is the true collapse load or is smaller than this load. This is true because of the fact that any smaller load will be in equilibrium with a safe statically admissible stress-system and therefore cannot be a collapse load. Note that, when using this theorem, we focus our attention entirely on the stresses. This was done also by Hornc [8], who dealt primarily with conditions on the stresses. The lower bound theorem provides a basic justification for this point of view.

We shall here use the yield function that was proposed by von Mises. It has the form (sec [5])

$$\Phi = \frac{1}{Y^2} \left[ \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2 \right].$$
Although we have eliminated above all considerations of compatibility of strains the problem is still so complicated that further simplifications must be made. Following the method of the approximate elastic solution, mentioned in the Introduction, this analysis is made two-dimensional in the $xz$ plane, so that $\sigma_x, \tau_{xz}$ and $\sigma_z$ are the only non-vanishing stresses. Moreover we assume the stresses to be independent of the $y$-coordinate. This implies that the condition of no surface traction on the inner sides of the flanges must be violated. The same error appears in the approximate elastic solution. This cannot, of course, be taken as a justification of the assumption in the problem of the plastic bending of the beam. Such justification must ultimately be found in a more complete theory or by experiments.

Two different stress fields are constructed in the following. In the first analysis (Section C) it is desirable to make a further approximation. This consists of assuming the $\sigma_z$ stress to vanish, and it turns out that one of the two equilibrium equations cannot generally be satisfied in this case. In the second analysis (Section D) the same assumption is made, but for this stress field it is found that a vanishing $\sigma_z$ does satisfy both equilibrium equations. Hence no approximation in this regard is involved in the second type of stress field. As this second analysis gives quite satisfactory values of the estimates of the limit moment no attempt is made to consider the effects of the $\sigma_z$ stress in the first analysis.
The first analysis is built up as a continuation of the conventional elastic theory of the bending of beams. In this theory it is known that the most highly stressed points of long beams are located in the outer fibers at the built-in section. Hence we assume that plastic regions start at these points and then spread in through the flange into the web, Fig. 6. At some stage a new plastic region will form starting from the center point of the web at the built-in section. In this first analysis we estimate the limit load by the load that corresponds to the stage when the stresses at the center of the beam at the built-in section just start to produce plastic flow. The present estimates of limit loads are conservative on this account. The calculation in the first part of the analysis considers this case where the plastic regions have the shape indicated in Fig. 6. However, when the yield condition is checked in the elastic region, we find that a critical point exists at the flange-web junction for beams of relatively short length. This suggests that a second plastic region will occur in the outer portions of the web, as shown in Fig. 7. The second part of this analysis considers this case. In the third part, beams of still shorter lengths are treated, namely the case when the second plastic region extends into a cross-section of the beam that otherwise would have been entirely elastic. This is shown in Fig. 8.

At the start of the second analysis (Section D) a stress field is constructed which is anticipated by the failure of a very short beam due to shear in the web. It is then found possible to generalize this stress field so that beams of arbitrary
lengths can be treated.

The second type of stress field is both simpler and more general than the first type, and it will probably provide the more useful basis for estimating limit moments in the presence of shear forces. On the other hand the second type of stress field is quite artificial, whereas the first type is a natural extension of the well confirmed simple theory of elastic-plastic bending of beams. In this respect it corresponds to and supplements the analysis of Horne [8].

C. Analysis for the first type of stress field.

For the first type of stress field this analysis is divided into three parts, based in turn on the three configurations of plastic regions shown in Figs. 6 - 8. The analysis will show that there exist for a given cross-section shape three decisive lengths $l_o$, $l_1$ and $l_2$ which determine the appropriate configuration.

1. Beams of length so that $l \geq l_1$ and $l \geq l_o$.

We here consider plastic regions as shown in Fig. 6. They are bounded by the outer sides of the flanges, by the built-in section and the curve $u(x)$. In zone BC of the beam the plastic regions extend into the web, but in zone CF the boundaries of the plastic regions are located in the flange. With the assumption $\sigma_z = 0$, the two remaining stresses in the plastic regions are described by the equilibrium equation and by the yield condition:
The yield condition (2) is identically satisfied if we relate the stresses to each other through the function \( a(x,z) \) in the following manner.

\[
\sigma_x = Y \cos a, \quad \tau_{xz} = \frac{Y}{\sqrt{3}} \sin a.
\]

The equilibrium equation becomes

\[
\sqrt{3} \left( \frac{\partial a}{\partial x} \sin a - \frac{\partial a}{\partial z} \cos a \right) = 0.
\]

The characteristics of this equation are lines along which \( a \) is constant. The expression

\[
d\alpha = \frac{\partial a}{\partial x} \, dx + \frac{\partial a}{\partial z} \, dz = 0
\]

together with equation (3) implies that the characteristics are straight lines with the slope

\[
\frac{dz}{dx} = -\frac{1}{\sqrt{3}} \cot a.
\]

On the boundary \( z = a \) we have \( \sigma_x = Y, \, \tau_{xz} = 0 \) or \( a(x,a) = 0 \).

The characteristics are therefore lines parallel to the \( z \)-axis and the stresses in the plastic region are given by

\[
\sigma_x = Y, \quad \tau_{xz} = 0.
\]

In the elastic part of the beam we follow the theory based upon the Bernoulli assumption. This implies a tensile stress linear in \( z \). In the elastic part of zone BC we therefore
have

\[ \sigma_x = \frac{a}{u} Y. \]

(From here on we consider only positive \( z \)-values. The quantities in the beam are either symmetrical or anti-symmetrical around the middle surface.) The equilibrium equation (1) determines the shear stress. Using the boundary condition \( \tau_{xz}(x,u) = 0 \) we obtain

\[ \tau_{xz} = -\frac{u^1}{2} \left(1 - \frac{Z^2}{u^2}\right) Y \]

where the prime denotes differentiation with respect to \( x \). The boundary \( u(x) \) between the plastic and elastic regions in the zone BC is determined by the total moment at any cross-section. It is

\[ P_x = M_p - \frac{1}{3} \frac{u^2}{c^2} M_{pw} \]

or

\[ \frac{u^2}{c^2} = 3 \frac{M_p - P_x}{M_{pw}} . \] (5)

The cross-section \( C \) is determined by \( u_C = c \) or

\[ x_C = \frac{M_p - \frac{1}{3} M_{pw}}{P} . \] (6)

In the elastic part of zone CF we have

\[ \sigma_x = \frac{z}{u} Y \]

and the shear stress in the flange is also given by equation (1)

\[ \tau_{xz} = -\frac{u^1}{2} \left(1 - \frac{Z^2}{u^2}\right) Y. \]

To find the shear stress in the web we must consider the difference in the width of the beam at the flange-web junction. We
let the equilibrium condition be satisfied only approximately; namely we assume that the shear stress has the same jump at the flange-web junction as the width of the beam. Hence the boundary condition for the shear stress in the web is

\[ \tau_{xz}(x, c) = -\frac{b_1}{b} \frac{u^4}{u^2} (1 - \frac{c^2}{u^2}) Y. \]

The equilibrium equation (1) then gives the shear stress in the web

\[ \tau_{xz} = -\frac{u^4}{2} [1 - \frac{c^2}{u^2} + \frac{b_1}{b} - 1](1 - \frac{c^2}{u^2}) Y. \]

As before the plastic-elastic boundary \( u(x) \) in zone CF is determined by the total moment at any cross-section. It is

\[ P_x = \begin{cases} (a^2 - u^2) b_1 + (u^2 - c^2) \frac{u}{u} b_1 \\
+ \frac{(u - c)^2}{u} b_1 [c + \frac{2}{3}(u - c)] + \frac{2}{3} \frac{c^3}{u} b_1 \end{cases} Y. \]

or

\[ \frac{u^3}{3} - 3 \left[ \frac{a^2}{c^2} - \frac{P_x b_1}{M_p w b_1} \right] b_1 + 2(1 - \frac{b}{b_1}) = 0. \]  \hspace{1cm} (7)

The cross-section \( F \) is determined by \( u_F = a \), i.e.,

\[ \frac{a^3}{c^3} - 3 \left[ \frac{a^2}{c^2} - \frac{P_{x_F} b_1}{M_p w b_1} \right] b_1 + 2(1 - \frac{b}{b_1}) = 0 \]

or

\[ x_F = \frac{J_1}{a_F} \]  \hspace{1cm} (8)

where \( I \) is the moment of inertia of the cross-section

\[ I = \frac{2}{3} [b_1 a^3 - (b_1 - b)c^3]. \]

In the entirely elastic zone \( FH \) the tensile stress is given by
The equilibrium equation (1) again determines the shear stress, so that

\[ \tau_{xz} = \frac{P(a^2 - z^2)}{2I} \quad \text{in the flange} \]

\[ \tau_{xz} = \frac{P(M_p - \frac{M}{I})c^2 - z^2}{2I} \quad \text{in the web}. \]

The validity of the stress-system, obtained above, depends on two conditions. First, it is assumed in Fig. 6 that the plastic regions extend into the web at the built-in section. Secondly, the yield function is assumed not to exceed unity in the elastic region. The first condition implies \( u_B \leq c \), i.e.,

\[ l \geq \frac{M_p - \frac{1}{3}M_{pw}}{P}. \]

For the second condition we form the yield function

\[ \Phi = (\sigma_x^2 + 3\tau_{xz}^2) \frac{1}{\chi^2} \]

in the different parts of the elastic region. We obtain

\[ \Phi_{BC} = \frac{a^2}{u^2} + \frac{27}{16} \left( \frac{P}{M_{pw}} \right)^2 \frac{c^2}{u^2} \left( 1 - \frac{z^2}{u^2} \right)^2 \]

\[ \Phi_{CF} = \frac{z^2}{u^2} + \frac{27}{16} \left( \frac{P}{M_{pw}} \right)^2 \frac{(u/c)^2 - (z/c)^2}{[(u/c)^3 - 1]b_1/b + 1} \quad \text{in the flange} \]

\[ \Phi_{CF} = \frac{z^2}{u^2} + \frac{27}{16} \left( \frac{P}{M_{pw}} \right)^2 \frac{1 - (z/c)^2 + [(u/c)^2 - 1]b_1/b}{[(u/c)^3 - 1]b_1/b + 1} \quad \text{in the web} \]
\[ \Phi_{PH} = \frac{P^2}{I^2Y^2} [x^2z^2 + \frac{3}{4}(a^2 - z^2)^2] \text{ in the flange} \]

\[ \Phi_{PH} = \frac{P^2}{I^2Y^2} [x^2z^2 + \frac{3}{4}\left(\frac{M_p}{M_{pw}}\right) c^2 - z^2)^2] \text{ in the web.} \]

When we look for possible maxima of \( \Phi \) we notice that in the zone FH \( \Phi \) has its maximum value at the section F. Hence we need only to check the zones BC and CF.

In the zone BC it is convenient to consider \( z^2/u^2 = \xi \) and \( 27/16 \left(\frac{P_c}{M_{pw}}\right)^2 c^2/u^2 = \eta \) as the two independent variables. Then

\[ \Phi_{BC}(\xi, \eta) = \xi + \eta(1 - \xi)^2. \]

A maximum appears on the boundaries \( \eta = \eta_{max} \) and \( \xi = 1 \). At \( \xi = 1 \) we have \( \Phi = 1 \), so the critical section is \( \eta = \eta_{max} \) corresponding to the smallest possible value of \( u \), i.e., the built-in section. The maximum of \( \Phi \) at this section appears for \( z = 0 \).

As mentioned in the previous section we take the load \( P_1 \) as an estimate of the limit load in this analysis, where \( P_1 \) is determined by the yield condition at this critical point. Hence we have

\[ \frac{27}{16} \left(\frac{P_1}{M_{pw}}\right)^2 \frac{c^2}{u_B^2} = 1. \]

\( u_B \) is eliminated by means of equation (5). This gives

\[ \frac{P_1 l}{M_p} = \frac{8}{9} \frac{c^2}{l^2} \frac{M_{pw}}{M_p} \left[ \sqrt{1 + \frac{9}{4} \left(\frac{c^2}{l^2} \frac{M_p}{M_{pw}}\right)} - 1 \right]. \quad (10) \]

The yield condition remains to be checked in zone CF.

We find when looking for extreme values that there are no
stationary points in the flange. On the boundaries of the elastic region in the flange we have on one side the plastic region and on the other side the junction to the web. Because of the jump of the shear stresses at this junction the stresses in the web are more critical. The yield function in the web also has maximum on the boundaries, i.e., \( z = 0 \) and \( z = c \). At \( z = 0 \) we have

\[
\Phi_{CF} = \frac{27}{16} \left( \frac{Pc}{M_{pw}} \right)^2 \frac{[(u/c)^2 - 1]b_1/b + 1}{[(u/c)^3 - 1]b_1/b + 1},
\]

As \( u/c \geq 1 \) \( \Phi_{CF} \) has maximum for smallest possible \( u/c \), i.e., \( u = c \). But for this value we know from zone BC that the yield condition is satisfied. Hence we have to check at the flange-web junction. We have at \( z = c \)

\[
\Phi_{CF} = \frac{c^2}{u^2} + \frac{27}{16} \left( \frac{Pc}{M_{pw}} \right)^2 \left( \frac{[(u/c)^2 - 1]b_1/b}{[(u/c)^3 - 1]b_1/b + 1} \right)^2.
\]

Equating \( \Phi_{CF} = 1 \) yields the following equation in \( u/c \):

\[
\left( \frac{u}{c} \right)^3 - 1 + \frac{b}{b_1} = \frac{3 \sqrt{3}}{4} \frac{Pc}{M_{pw}} \frac{u}{c} \sqrt{\frac{u^2}{c^2} - 1} \tag{11}
\]

or

\[
\left( \frac{u}{c} \right)^6 - \frac{27}{16} \left( \frac{Pc}{M_{pw}} \right)^2 \left( \frac{u}{c} \right)^4 - 2(1 - \frac{b}{b_1}) \left( \frac{u}{c} \right)^3 + \frac{27}{16} \left( \frac{Pc}{M_{pw}} \right)^2 \left( \frac{u}{c} \right)^2 + (1 - \frac{b}{b_1})^2 = 0.
\]

The last equation shows that there are two or zero positive roots. We can get an approximate solution by considering the quantity \( u/c - 1 = \epsilon \), which is assumed to be a small number as
compared to unity. The equation becomes

$$\sqrt{\varepsilon}[\frac{3\sqrt{3}}{4} \frac{Pc}{M_{pw}} (1 + \varepsilon) \sqrt{\varepsilon} + 2 - \sqrt{\varepsilon} (3 + 3\varepsilon + \varepsilon^2) = \frac{b}{b_1}].$$

As $\frac{1}{4} \sqrt{3} \frac{Pc}{M_{pw}}$ may be of the same order as $\sqrt{\varepsilon}$, we get in approximate form

$$\sqrt{\varepsilon} (\frac{3\sqrt{6}}{4} \frac{Pc}{M_{pw}v} - 3 \sqrt{\varepsilon}) = \frac{b}{b_1}$$

or

$$\varepsilon = \frac{1}{3} \left[ \frac{9}{16} (\frac{Pc}{M_{pw}})^2 - \frac{b}{b_1} + \sqrt{\frac{9}{16} (\frac{Pc}{M_{pw}})^2 - \frac{b}{b_1}}^2 - (\frac{b}{b_1})^2 \right].$$

The critical value, when yield just starts at one point, corresponds to vanishing of the radical, i.e.,

$$\frac{Pc}{M_{pw}} = \frac{1}{3} \frac{2b}{b_1} = \frac{b}{3b_1},$$

located at the section where

$$\varepsilon = \frac{b}{c} - 1 = \frac{1}{3} \frac{b}{b_1}.$$

In the cases of I-sections of particular interest here $b/b_1 \sim 0.1$, so that the assumption that $\varepsilon$ is small as compared to unity is satisfied.

We now consider the beam loaded with the force $P_1$ according to equation (10). Then elimination of $P_1$ between (10) and (13) gives an approximate value of the length of the beam, $l_1$, for which the two critical points start to yield simultaneously. We obtain

$$\frac{l_1}{c} = 3 \sqrt{\frac{2}{3}} \sqrt{\frac{b_1}{b} (\frac{M_{pw}}{P_M} - \frac{2b}{b_1})}.$$
On the other hand eliminating $P_1$ between the inequality (9), taken as an equation, and equation (10) gives

$$\frac{l_0}{c} = \sqrt{\frac{3}{4}} \left( 3 \frac{M_p}{M_{pw}} - 1 \right).$$

(15)

Hence we have the following two conditions for the validity of this first part of the analysis:

$$\frac{l}{c} \geq \frac{l_1}{c} \quad \text{and} \quad \frac{l}{c} \geq \frac{l_0}{c}.$$

It is interesting to compare these two values for standard I-beams. Table 1 shows that for many common sections $l_1 > l_0$. This shows that the stresses at the flange-web junction play a decisive role even for relatively long beams.

The influence of the shear stresses upon the limit moment is expressed by equation (10). This influence is shown in Fig. 9, where $P_1 \ell/M_p$ is plotted as function of $\frac{l}{c} \sqrt{M_{pw}/M_p}$.

2. Beams of length so that $l_1 > \ell > l_2$ and $\ell > l_0$.

We now consider beams that have a length shorter than $l_1$. We retain the previous estimate of the limit load as the force $P_1$ according to equation (10). The stress field must therefore be modified so that the yield condition $\Phi < 1$ is satisfied everywhere. We introduce second plastic regions in the web as shown in Fig. 7. These regions are bounded by the curve $v(x)$ and occupy the zone DE, where we assume, in this part of the analysis, that the cross-sections D and E are located between the cross-sections C and F.
There is no reason to change the previous stresses in the zones BD and EH of the beam. The cross-sections D and E are determined by equation (11) or approximately by equation (12). The condition that the cross-section E will be located to the left of cross-section F is given by $u_E \leq a$. By means of equations (8) and (11) this condition is

$$\frac{L}{c} \geq \frac{L_2}{c} = \sqrt{\frac{3}{4}} [K \frac{M_p}{M_{pw}} - \sqrt{\frac{3}{K}}]$$

(15)

where

$$K = \frac{2a}{I_Y} \sqrt{\frac{b_1}{b} \frac{M_p}{M_{pw} M_{pf}}}.$$

We also assume, as before, that $u_B \leq c$. The bounds for the length of the beam for which this part of the analysis applies, therefore, are

$$\frac{L_1}{c} \geq \frac{L}{c} \geq \frac{L_2}{c} \quad \text{and} \quad \frac{L}{c} \geq \frac{L_0}{c}.$$

Now consider zone DE. Both the flange and the web are partly elastic, partly plastic. In the plastic region of the flange we have as before $\sigma_x = Y$, $\tau_{xz} = 0$. In the plastic region of the web we again relate the stresses to each other by $\sigma_x = Y \cos \alpha(x,z)$ and $\tau_{xz} = \frac{Y}{\sqrt{3}} \sin \alpha(x,z)$. If the boundary condition at the flange-web junction were given, we then could determine the stresses along the characteristics. Therefore we consider $\alpha(x,c) = f(x)$ as a given function and express further quantities in terms of it.

In the elastic region of the flange the assumption of linear variation of $\sigma_x$ and of continuity of $\sigma_x$ and $\tau_{xz}$ across the elastic plastic interface then yields
\[ \sigma_x = \frac{(u - z) \cos f + z - c}{u - c} Y. \]

The equilibrium equation (1) together with the boundary condition \( \tau_{xz}(x, u) = 0 \) gives

\[ \tau_{xz} = -\frac{1}{2} \frac{\partial}{\partial x} \left[ \frac{(u - z)^2}{u - c} (1 - \cos f) \right] Y. \]

The other boundary condition

\[ \tau_{xz}(x, c) = \frac{Yb}{\sqrt{3} b_1} \sin f \]

determines the function \( u(x) \) to within an arbitrary constant \( A \), as follows:

\[ (u - c)(1 - \cos f) = A + \frac{2b}{\sqrt{3} b_1} \int_{x}^{x_D} \sin f \, dx. \]

The boundary condition \( u = u_D \) at cross-section \( D \) determines \( A \) so that

\[ u = c + \frac{(u_D - c)(1 - \cos f(x_D))}{1 - \cos f} + \frac{2b}{\sqrt{3} b_1 (1 - \cos f)} \int_{x}^{x_D} \sin f \, dx. \tag{16} \]

But \( u \) is given also at section \( E \). Hence we must impose one condition on the function \( f \), namely

\[ \frac{\sqrt{3} b_1}{2b} \left[ (u_E - c)(1 - \cos f(x_E)) - (u_D - c)(1 - \cos f(x_D)) \right] = \int_{x}^{x_D} \sin f \, dx. \tag{17} \]

This equation can also be derived as the equilibrium equation of forces in the \( x \) direction on the segment at the flange between \( D \) and \( E \).

In the second plastic region in the web the stresses are given along each characteristic. Hence \( \sigma_x = Y \cos f(x) \) and
\( \tau_{xz} = \frac{y}{\sqrt{3}} \sin f(x) \) where the value of \( x \) refers to the coordinate of the characteristic at the flange-web junction, Fig. 10. As before, the characteristics are straight lines in this case making the angle \( \varphi(x) \) to the z-direction. In view of equation (4)

\[
- \frac{dx}{dz} = \sqrt{3} \tan f = \tan \varphi.
\]  

(18)

It is found convenient to denote the length of the characteristic by \( g(x) \sqrt{1 + 2 \sin^2 f(x)} \).

Now consider a "cross-section" of the beam that follows the characteristic in the second plastic region as shown in Fig. 10. Due to the slope of the characteristic the center part of this "cross-section" has a coordinate, denoted by \( s \), that differs from the coordinate \( x \) of the outer part of the "cross-section". The coordinates are related by

\[
s(x) = x + g(x) \sqrt{1 + 2 \sin^2 f(x)} \sin \varphi(x)
\]

\[= x + \sqrt{3} g(x) \sin f(x).\]

The function \( v \) is now determined at \( s \) by

\[
v[s(x)] = c - g(x) \sqrt{1 + 2 \sin^2 f(x)} \cos \varphi(x)
\]

\[= c - g(x) \cos f(x).\]

Hence the assumptions of continuity and of linear variation of \( \frac{d}{dx} \) gives

\[
\frac{d}{dx}[s(x),z] = \frac{z \cos f(x)}{c - g(x) \cos f(x)} Y.
\]
The equilibrium equation is

$$\frac{\partial \sigma}{\partial x} \frac{dx}{ds} + \frac{\partial \tau}{\partial z} = 0.$$ 

This determines $\tau_{xz}$ together with the boundary condition

$$\tau_{xz}[s(x),c - g \cos f(x)] = \frac{y}{\sqrt{3}} \sin f(x).$$

We obtain

$$\tau_{xz} = \frac{1}{2} \left[ \frac{2}{\sqrt{3}} \sin f \right. \right.$$

$$+ \left( 1 - \frac{\varepsilon^2}{\sqrt{3}} \right) \frac{g' \cos^2 f - c' \sin f}{(c - g \cos f)^2 \left( 1 + \frac{1}{\sqrt{3}} (g \sin f)' \right)}.$$ 

Now it remains to determine $g(x)$ in terms of $f(x)$. As before we set up the equation of total moment, now taken with respect to the point $(s,0)$. The external moment is given by equation (10) as

$$P_1 s = \frac{8}{9} \frac{ts}{c^2} M_{pw} \left( \sqrt{1 + \frac{3}{4} \frac{c^2 M_p}{M_{pw}}} - 1 \right).$$

The plastic regions of the flanges contribute $M_1$

$$M_1 = (a^2 - u^2)b_1 Y.$$ 

The elastic regions of the flanges contribute the moment $M_2$ which is made up of two parts $M_2^{(1)}$ and $M_2^{(2)}$ arising from the tensile and shear stresses, respectively

$$M_2^{(1)} = \frac{1}{3} (u - c)[2u + c + (u + 2c) \cos f]b_1 Y$$

$$M_2^{(2)} = -\frac{1}{\sqrt{3}} g \sin f [(u - c)^2 (1 - \cos f)]' b_1 Y.$$
The plastic regions of the web contribute $M_3$ which also consists of two parts, $M_3^{(1)}$ and $M_3^{(2)}$, from the two stresses, $\sigma_x$ and $\tau_{xz}$ respectively.

\[
M_3^{(1)} = (2c - g \cos f) g \cos^2 f bY
\]
\[
M_3^{(2)} = 2gc \sin^2 f bY
\]
\[
M_3 = (2gc - g^2 \cos^3 f) bY.
\]

Finally the elastic part of the web contributes $M_4$

\[
M_4 = \frac{2}{3} (c - g \cos f)^2 \cos f bY.
\]

The moment equilibrium equation then becomes

\[
P_{es} = M_1 + M_2 + M_3 + M_4.
\]

It is quadratic in $g(x)$. We therefore obtain

\[
g(x) = A(x) - \sqrt{A^2(x) - B(x)}
\]

where

\[
A(x) = \frac{c - \frac{3\sqrt{3}}{2} \frac{P_1c^2}{M_{pw}} \sin f + \left[2u + \frac{\sqrt{3} b_1}{2b}(u - c)^2 f' \right] \sin^2 f}{\cos^3 f}
\]

\[
B(x) = \frac{3c^2 \frac{P_1x - M_{pf}}{M_{pw}} - 2c^2 \cos f + \frac{b_1}{b}(u - c)(u + 2c)(1 - \cos f)}{\cos^3 f}
\]
To complete the analysis we must write conditions ensuring continuity of stresses at cross-sections D and E. The following requirements are imposed on the values of $f(x)$ and $f'(x)$ at D and E:

$$
\begin{align*}
  f(x_D) &= \cos^{-1} \left( \frac{c}{u_D} \right); & f'(x_D) &= \frac{u_D c}{u_D \sqrt{u_D^2 - c^2}}. & (22) \\
  f(x_E) &= \cos^{-1} \left( \frac{c}{u_E} \right); & f'(x_E) &= \frac{u_E c}{u_E \sqrt{u_E^2 - c^2}}. & (23)
\end{align*}
$$

The previous condition on $f(x)$, (17), now takes the form

$$
\frac{\sqrt{3}}{2b} \frac{b_1}{(u_E - c)^2 - (u_D - c)^2} = \int_{x_D}^{x_E} \sin f \, dx. & (24)
$$

Any function $f(x)$ that fulfills these five boundary conditions, equations (22), (23) and (24), and produces plastic regions of the type assumed in Fig. 7 will define a statically admissible system of stresses within the framework of the assumptions we have made. In the numerical example one such function $f(x)$ is chosen.

3. Beams of length so that $l_2 \geq l \geq l_0$.

The beams treated in this part have such length and cross-section dimensions that the plastic region that started from the outer fiber extends into the web, and so that the second plastic region extends into the zone FH, see Fig. 8. The length of the beam must therefore fulfill
One further condition should be mentioned namely the requirement that the cross-section $G$ does not fall near the free end of the beam.

The same stress field as before is used in the zones $BF$ and $GH$. The "cross-section" $F$ is given by $u_F = a$, i.e.,

$$(a - c)(1 - \cos f(x_F)) = \frac{(u_D - c)^2}{u_D} + \frac{2b}{\sqrt{3}} b_1 \int_{x_F}^{X_D} \sin f \, dx. \quad (25)$$

The cross-section $G$ is determined by the yield condition in the web at $z = c$. It is

$$1 = \frac{p_1 c^2}{I Y} \left[ x_G^2 + \frac{3}{4} c^2 \left( \frac{M_{PF}}{M_{PW}} \right)^2 \right]$$

so

$$\frac{x_G}{c} = \sqrt{\frac{I Y}{p_1 c^4} - \frac{3}{4} \left( \frac{M_{PF}}{M_{PW}} \right)^2}. \quad (26)$$

In the zone $FG$ the flange is entirely elastic, i.e., the web contains the second plastic region, Fig. 8. The stresses in the web are determined as before by the function $f(x)$. The expressions for the stresses in the web remain unchanged, so that in the plastic region of the web along each characteristic, Fig. 11

$$\sigma_x = Y \cos f, \quad \tau_{xz} = \frac{Y}{\sqrt{3}} \sin f$$

and in the elastic region of the web
In the flange, which is in the elastic state, we let \( \sigma_x \) have linear variation in \( z \) and be continuous at the flange-web junction. This gives

\[
\sigma_x = \frac{(a - z) \cos f + (z - c)h}{a - c} Y
\]

where we denote \( \sigma_x(x,a) = Yh(x) \). The function \( h(x) \) will be found in terms of \( f(x) \). The equilibrium equation (1) together with the boundary condition \( \tau_{xz}(x,a) = 0 \) determines

\[
\tau_{xz} = \frac{a - z}{2(a - c)} [(a + z - 2c)h' - (a - z)f' \sin f] Y.
\]

The boundary condition for \( \tau_{xz} \) at the flange-web junction determines \( h(x) \). We have in the flange

\[
\tau_{xz}(x,c) = \frac{1}{2} (a - c)(h' - f' \sin f) Y = \frac{b Y}{\sqrt{3}} \sin f.
\]

Hence

\[
h = B - \cos f + \frac{2b}{\sqrt{3} b_1(a - c)} \int_{X_G}^{X} \sin f \, dx.
\]

The constant \( B \) is determined from the condition of continuity of \( \sigma_x \) at the section \( G \). We obtain

\[
h = (\hat{\alpha} + 1) \sqrt{1 - \frac{3}{4} \left( \frac{1}{IY M_{pw}} \right)^2 \cos f + \frac{2b}{\sqrt{3} b_1(a - c)} \int_{X_G}^{X} \sin f \, dx}.
\]
But the tensile stress is also given at the section F. This imposes the following condition on \( f(x) \):

\[
\frac{\sqrt{3} b_1}{2b} [(a - c)(1 - \cos f(x_F))] - \frac{a^2 - c^2}{c} \sqrt{1 - \frac{3}{4} \left( \frac{P_1 c^2 M}{I Y^2} \right)^2} \int_{X_0}^{X_F} \sin f \, dx. \tag{28}
\]

This condition is equivalent to equation (17) applying to zone DE. It also expresses the requirement of the equilibrium of the forces in the x-direction for the segment of the flange between F and G.

We now determine the function \( g(x) \) by the overall moment equilibrium equation with respect to the point \((s, 0)\), Fig. 11. As before, the external moment is

\[
P_1 s = \frac{8}{9} \frac{I Y}{c^2} M \left( \frac{1}{\sqrt{1 + \frac{Q c^2 M}{I Y^2}}} - 1 \right).
\]

The contributions to the moment from stresses in the web are as given in the previous section, namely

\[
M_3 = (2gc - g^2 \cos^3 f) bY
\]

\[
M_4 = \frac{2}{3} (c - g \cos f)^2 \cos f bY.
\]

The flange contributes the moment \( M_2 \), which consists of two parts, \( M_2^{(1)} \) and \( M_2^{(2)} \), arising from the tensile stresses and the shear stresses, respectively:
As before the moment equilibrium equation

\[ P_1 s = M_2 + M_3 + M_4 \]

determines \( g(x) \) so that

\[ g(x) = A - \sqrt{A^2 - B} \quad (29) \]

where

\[ c = \frac{3\sqrt{3}}{2} \frac{P_c}{M_{pw}} \sin f + \frac{2a + \sqrt{3} b_1}{2b} (a - c)^2 f' \sin f \]

\[ A(x) = \frac{\cos^3 f}{3c^2 \frac{P_c x}{M_{pw}} - 2c^2 \cos f - \frac{b_1}{b}(a-c)[(a+2c)\cos f+(2a+c)h]} \cos^3 f \quad (30) \]

\[ B(x) = \frac{\frac{P_c x}{M_{pw}} - 2c^2 \cos f - \frac{b_1}{b}(a-c)[(a+2c)\cos f+(2a+c)h]} \cos^3 f \quad (31) \]

As before, we must check that the stresses are continuous at the cross-sections F and G. The stresses at section F are given on both sides of the cross-section in terms of \( f(x) \). Hence \( f(x) \) must be continuous and have a continuous first derivative at \( x_F \). Continuity of the shear stresses at F imposes one more condition, namely \( u_F' = 0 \). In terms of \( f(x) \) this condition takes the form
\[ f'(x_F) = -\frac{2}{\sqrt{3}} \frac{a + c}{c^2} M_{pw} \]  

At section G the continuity of stresses requires
\[ f(x_G) = \sin^{-1} \frac{\sqrt{3} P_1 e^{2M_{pw}}}{2\sqrt{3} M_{pw}}; \quad f'(x_G) = -\frac{2M_{pw}}{\sqrt{3} c M_{pw}}. \]  

Any function \( f(x) \) that fulfills the four conditions (28), (32) and (33) and produces stresses of the type assumed in Fig. 8 will, within our assumptions, produce a statically admissible stress field. Note that the position of section F and the value of the function \( f(x) \) at F are not determined by the stress conditions. These values may be chosen so that the numerical calculations become as simple as possible. It will be convenient to assign a value for \( x_F \) at the very beginning of the computation. Thereafter we construct a stress field in the zone FG by using an \( f(x) \) that fulfills the conditions (28), (32) and (33). We then finish the problem in dealing with the zone DF, i.e., we choose an \( f(x) \) that fulfills conditions (22), (24) and the values of \( f(x_F) \) and \( f'(x_F) \) that are obtained from zone FG. The third numerical example in the following section is computed according to this scheme.


In this section an outline is shown of the computations that determine the shape of the plastic regions for three special cases all applied to an 8 WF 40 beam. The numerical values of the different quantities are given to as many as six significant figures, although of course for practical problems such precision
is meaningless. It is necessary in certain cases to carry this number of significant figures in order to show some of the properties of the analysis.

We have chosen in our examples one of the beams listed in Table 1. This table shows the computed lengths \( l_0, l_1 \) and \( l_2 \) for some beam sections which have been subjected to laboratory tests [11], [12], [13], and for a few other common beam sections. From these beams the 8 WF 40 section has been chosen for detailed discussion because it has a large value of the ratios \( M_p/M_{pw} \) and \( b_1/b \). The cross-section dimensions are given in Table 1. We compute three examples, choosing the length of the beam so that one example is provided for each part of the analysis.

In the first example we consider the shortest possible 8 WF 40 beam in which the second plastic region does not appear. Table 1 gives \( l_1/c = 21.1 \), and we therefore take \( l/c = 21.1 \) for the first example. The estimate for the limit load in this case is given by equation (10) as \( P_{1}/M_p = 0.9895 \). The beam contains the zones BC, CF and FH where \( x_C/c = 20.48 \), (Eq. 6), and \( x_F/c = 19.02 \), (Eq. 8). The elastic-plastic interface is computed in the zones BC and CF according to equations (5) and (7). The results are shown in Table 2 and in Fig. 12.

In the second example we choose the length \( l/c \) so that \( l_1/c \leq l/c \leq l_2/c \). We take \( l/c = 12.5356 \). (The actual choice was to make \( u_B/c = 0.85 \) exactly.) The estimate for the limit load is given by equation (10) as \( P_{1}/M_p = 0.9715 \). The beam is divided into the zones BC, CD, DE, EF and FH according to Fig. 7.
The elastic-plastic interface in zone BC is given by equation (5). The results are shown in Table 3. Equation (6) gives $x_c/c = 12.3942$. The cross-sections $x_D/c$ and $x_E/c$ are determined from $u_D/c$ and $u_E/c$ in the following way. Equation (12) determines approximately

$$u_D/c - 1 = 0.00176; \quad u_E/c - 1 = 0.129.$$  

Using equation (11) we obtain the more accurate values

$$u_D/c - 1 = 0.00175; \quad u_E/c - 1 = 0.14256.$$  

Equation (7) then gives

$$x_D/c = 12.3921; \quad x_E/c = 11.6367.$$  

Cross-section F is determined by equation (8) as

$$x_F/c = 11.5074.$$  

The zones CD and EF become so narrow that the boundary values of $u/c$ are sufficient to determine the elastic-plastic interface in these zones. We therefore now focus our attention on zone DE.

In order to construct a solution in zone DE we need to determine the function $f(x)$. This is restricted only by the boundary conditions, which are

$$f(12.3921c) = 0.05917; \quad f'(12.3921c) = -14.881/c$$  

$$f(11.6367c) = 0.50488; \quad f'(11.6367c) = -0.17083/c$$  

and
12.3921c
\[ \sin f \, dx = 0.34083c. \quad (24a) \]
11.6367c

We must therefore choose a certain form of the function \( f(x) \), this choice containing five arbitrary constants so that the above conditions can be satisfied. Note that even if the above conditions are satisfied the choice of \( f(x) \) may not be satisfactory due to the fact that the results may not agree with the general type of solution assumed in Fig. 7. E.g., the function \( v(x) \) may become parallel at a point with the \( z \)-axis or it may become larger than \( c \) at some point. Both cases must be avoided.

For the choice of \( f(x) \) in this case we notice that \( f'(x_D) \) is numerically large. This can be taken account of conveniently by choosing the \( f(x) \) basically as a parabola with the axis parallel to the \( x \)-axis and the vertex near \( x_D \). This suggests the introduction of the new variable \( y \) and also of the square root as follows:

\[ y = 12.3921 + k - \frac{x}{c} \]
and
\[ f(y) = A + B \sqrt{y} + Cy + Dy^2. \]

The five constants \( k, A, B, C \) and \( D \) are to be determined. The conditions become now

\[ f(k) = 0.05917; \quad f'(k) = 14.881 \]

\[ f(k + 0.7554) = 0.50488; \quad f'(k + 0.7554) = 0.17083 \]

and
k+0.7554
\[ \sin f \, dy = 0.34083. \]

Prime denotes differentiation with respect to \( y \). These conditions give

\[ k = 0.0054; \quad A = -0.1180; \quad B = 2.6378; \]
\[ C = -3.0663; \quad D = 1.1344. \]

The function \( f \) is shown in Fig. 13. Equations (16), (18) and (19) determine \( u, \varphi \) and \( g \); see Table 4. The plastic regions are shown in Fig. 14.

As the third example consider the shortest possible length for which the previous analysis applies, i.e.,

\[ \lambda/c = \lambda_0/c = 10.53492. \]

Very similar considerations apply in this case as compared to the foregoing example. The presentation is therefore shortened. We have the zones CD, DF, FG and GH as in Fig. 8. In this case equation (10) gives \( P/l/M_p = 0.9605 \). Equation (12) gives \( u_D/c - 1 = 0.00119 \) and from equation (11)

\[ u_D/c - 1 = 0.001185. \]

Equations (7) and (26) give \( x_D/c = 10.53385 \) and \( x_G/c = 9.29002. \) Conditions (22), (32) and (33) are:

\[ f(10.53385c) = 0.04865; \quad f'(10.53385c) = -22.011/c \quad (22b) \]
\[ f'(x_F) = -0.33448/c \quad (32b) \]
\[ f(9.29002c) = 0.60659; \quad f'(9.29002c) = -0.15514/c. \quad (33b) \]

Conditions (24) and (28) give
\[ \int_{9.29002c}^{x_F} \sin f \, dx = 0.6835^4 \quad (24b) + (28b) \]
\[ \int_{9.29002c}^{x_F} \sin f \, dx = 2.9898 \cos f(x_p) - 0.77136. \quad (28b) \]

In this example the cross-section F is arbitrarily chosen. We take \( x_F = 9.5 \).

In zone FG we let

\[ f = 0.60659 - 0.15514y + Ay^2 + By^3 \]

where

\[ y = x/c - 9.29002. \]

The conditions (33b) are satisfied while A and B are determined by (28b) and (32b). The constants become \( A = 4.1964 \) and \( B = -14.679 \). The equations (18), (27) and (29) determine \( \varphi, h \) and \( g \) in zone FG, see Table 5.

In zone DF we have the following conditions on \( f(x) \)

\[ f(10.53385c) = 0.04865; \quad f'(10.53385c) = -22.011/c \]
\[ f(9.5c) = 0.62314; \quad f'(9.5c) = -0.33448/c \]

and

\[ \int_{9.5c}^{10.53385c} \sin f \, dx = 0.56189c. \]

Now choose

\[ f = A + B \sqrt{y} + Cy + Dy^6 \]

where
y = 10.53385 + k - x/c.

The above five conditions determine $k = 0.00171$, $A = -0.02903$, $B = 1.93656$, $C = -1.40448$ and $D = 0.11020$. Equations (16), (18) and (19) determine $u$, $\varphi$ and $g$ in zone DF, see Table 6. The function $f$ and the plastic regions for this third example are shown in Figs. 15 and 16.

D. A second example of an approximate statically admissible stress field.

The results of the previous analysis in this paper show that for beams that have a length ratio greater than $l_0/c$ the reduction of the limit moment is at most of order 10%. Examples of these reductions are given in Table 1 in the column that shows the value of $P_1/M_p$ for beams having the length ratio $l_0/c$. The cases where shear stresses become important therefore are expected to occur for beams that have length shorter than $l_0$. A considerable reduction in the maximum end load of a built-in I-beam may take place when the beam becomes so short that the shear capacity of the web is not sufficient to balance the longitudinal force in the flanges. A failure, as depicted in Fig. 17, may then happen. As cross-sections no longer remain plane and normal to the center-line we can assume that the web carries a small amount of the longitudinal stresses so that in the web the shear stresses nearly reach their yield-value. We shall now construct a statically admissible stress field for this case -- starting with the same assumptions as before -- where this stress field is based on the failure of the web in
pure shear. As depicted in Fig. 18 let the stress field be composed as follows:

\[
\sigma_x = 0; \quad \tau_{xz} = \frac{Y}{\sqrt{3}} \quad \text{in the web}
\]

\[
\sigma_x = \frac{b x Y}{\sqrt{3} b_1 (a - c)}; \quad \tau_{xz} = \frac{b (a - z) Y}{\sqrt{3} b_1 (a - c)} \quad \text{in the flange.}
\]

The equilibrium condition (1) is satisfied everywhere and the yield condition (2) is obviously satisfied in the web. At the flange-web junction the shear stresses are in equilibrium in the same sense as before, and the longitudinal stresses are discontinuous. Such a discontinuity is permissible in the construction of a statically admissible stress field, see [14]. This stress field is valid when the yield condition (2) is not violated in the flanges. The yield condition in the flange takes the form

\[
\left[ \frac{b x}{\sqrt{3} b_1 (a - c)} \right]^2 + 3 \left[ \frac{b (a - z)}{\sqrt{3} b_1 (a - c)} \right]^2 \leq 1.
\]

The left-hand side has its maximum value at \( x = \frac{1}{3} \) and \( z = c \).
The yield condition will therefore be satisfied for beams having a length \( \ell \leq \ell_3 \) where \( \ell_3 \) is obtained from the following equation:

\[
\left[ \frac{b \ell_3}{\sqrt{3} b_1 (a - c)} \right]^2 + 3 \left[ \frac{b}{\sqrt{3} b_1} \right]^2 = 1
\]

or

\[
\frac{\ell_3}{c} = \sqrt{3} \left( \frac{a}{c} - 1 \right) \sqrt{\frac{(b_1/b)^2}{b_1^2}} - 1. \quad (34)
\]

This length is computed and shown in Table 1. The estimate of the maximum total shear force, \( P_2 \), becomes in this case
\[ P_2 = 2bc \frac{V}{\sqrt{3}} + (a - c)b \frac{V}{\sqrt{3}} = \frac{1}{\sqrt{3}} (a + c)bY. \]

Therefore
\[ \frac{P_2}{M_p} = \frac{1}{\sqrt{3}} (a/c + 1) \frac{\ell}{c} \frac{M_{pw}}{M_p}. \tag{35} \]

The above stress field suggests a more general field as depicted in Fig. 19. We let
\[ \sigma_x = k_1 \frac{X}{c} Y; \quad \tau_{xz} = k_1 \frac{a - z}{c} Y \quad \text{in the flange} \]
\[ \sigma_x = k_2 \frac{X}{c} Y; \quad \tau_{xz} = k_2 \frac{c - z}{c} Y + k_1 \frac{b_1}{b} (a/c - 1)Y \quad \text{in the web}. \]

We assume that the two constants \( k_1 \) and \( k_2 \) are non-negative. The equilibrium condition (1) is satisfied everywhere in the same sense as before. The yield condition must be checked in the flange and the web. We let \( k_1 \) and \( k_2 \) be so determined that the yield condition is simultaneously satisfied at the most critical points both in the flange and in the web. In the flange this point is \( x = \ell \) and \( z = c \). The following equation therefore determines \( k_1 \):
\[ (k_1 \ell/c)^2 + 3 [k_1(a/c - 1)]^2 = 1 \]
\[ k_1 = \frac{1}{\sqrt{(\ell/c)^2 + 3(a/c - 1)^2}}. \]

In the web the critical point is \( x = \ell \) and \( z = 0 \). We therefore choose \( k_2 \) so that
\[ (k_2 \ell/c)^2 + 3[k_2 + k_1 b_1/b(a/c - 1)]^2 = 1 \]
\[ \frac{k_2}{k_1} = \frac{1}{\sqrt{A^2 + B - A}} \]
where
\[ A = 3 \frac{b_1/b}{(l/c)^2 + 3} \left(\frac{a/c - 1}{(l/c)^2 + 3}\right) \]
\[ B = \frac{(l/c)^2 - 3(a/c - 1)^2[(b_1/b)^2 - 1]}{(l/c)^2 + 3} \]

The condition \( k_1 > 0 \) is satisfied but the condition \( k_2 > 0 \) shows that \( B > 0 \), and this restricts the length of the beam. This restriction is

\[ \frac{l}{c} \geq \sqrt{3} (a/c - 1) \sqrt{(b_1/b)^2 - 1} = \frac{l}{c}. \]  

(36)

This stress field together with the previous one therefore applies to any length of the beam. The total shear force now becomes

\[ P_2 = k_1(a/c - 1)(a - c)b_1 Y + 2k_1(a/c - 1)b_1 cY + k_2 bcY. \]

Therefore

\[ \frac{P_2 l}{M_p} = \frac{M_{pw}}{M_p} c \left\{ k_1 [(a/c)^2 - 1] b_1/b + k_2 \right\}. \]  

(37)

Within our assumptions equations (35) and (37) provide a lower bound for the limit load. The curve of \( P_2 l/M_p \) versus \( l/c \) has been computed for the 8 WF 40 section. The result is shown in Table 7 and in Fig. 21.

In this stress field the assumption that \( \sigma_z \) vanishes is strictly valid. This is true because the shear stresses are independent of \( x \) so that the equilibrium equation involving \( \sigma_z \) is identically satisfied. The stress fields that we have constructed give a lower bound for the collapse load. In the next
section we shall obtain an upper bound which will be applicable for short beams.

E. A kinematically admissible velocity field for short beams.

We shall now make use of the second limit analysis theorem of Drucker, Prager and Greenberg [14], which states that a load obtained in a definite manner from a kinematically admissible velocity field is an upper bound for the collapse load. A velocity field is defined as kinematically admissible when it satisfies the imposed velocity conditions at the boundary of the body and when the velocities vanish on those parts of the boundary where the surface tractions are not prescribed. In this paper we consider only velocity fields which are continuous and satisfy the condition of incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where u, v and w are the velocity components in the x, y and z directions. The upper bound \(P^*\) of the load is obtained by equating the rate of work produced by the applied forces to the rate of plastic work in the body.

It seems intuitively likely that a reasonable upper bound of the collapse load should be such that the value \(P^*l/M_p = 1\) is exceeded by only a small amount, if at all. For moderately long I-beams we have only been able to construct velocity fields that give \(P^*l/M_p\) considerably larger than unity. We therefore confine attention to short beams where a "good" velocity field is easily obtained and restrict ourselves to the
length of the beams so that $P \cdot \ell / M_p \leq 1$. The velocity field that is chosen is depicted in Fig. 20. It is symmetric with respect to the middle surface OFG. Part ABFG is rigid and fixed. The part BCDE of the flange is rigid and rotates with the angular velocity $\omega$ about the fixed point B. In part ABC of the flange let

$$u = w = -\frac{1}{2} \omega (\ell - x - a + z); \quad v = 0.$$  

This is a motion parallel to AB, with velocities continuous on AB and BC. In part BEOF of the web let

$$u = v = 0; \quad w = -\omega (\ell - x - a + c).$$

The velocity is continuous on BE and the motion is parallel to BF. As before we use the von Mises yield condition and therefore also the von Mises flow rule. We have constructed a plane velocity field. The rate of plastic work, $W_p$, for such fields is derived in [5], page 214.

$$W_p = \frac{\nabla}{\sqrt{3}} \int \dot{\Gamma} dV$$

where $dV$ is the volume element and $\dot{\Gamma}$ is given by

$$\dot{\Gamma} = [(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2]^{1/2}.$$  

We obtain both in ABC and BEOF

$$\dot{\Gamma} = \dot{u}.$$  

The rate of plastic work therefore is

$$W_p = \frac{2}{\sqrt{3}} Y \omega [V_{ABC} + V_{BEOF}]$$
\[ W_p = \frac{2}{\sqrt{3}} M_{pw} \omega [b_1/b(a/c - 1)^2 + \ell/c - a/c + 1]. \]

The rate of external work is \( W_e \)

\[ W_e = P*\omega(\ell - a + c). \]

The limit analysis theorem therefore gives the following upper bound for the limit moment

\[ \frac{P*\ell}{M_p} = \frac{2}{\sqrt{3}} \frac{M_{pw}}{M_p} \left[ \frac{b_1/b(a/c - 1)^2}{\ell/c - a/c + 1} + 1 \right] \]

(38)

\( P*\ell/M_p \) becomes unity for

\[ \ell/c = A + \sqrt{A^2 - B} \]

where

\[ A = \frac{1}{4} \left[ \sqrt{3} \frac{M_p}{M_{pw}} - 2b_1/b (a/c - 1)^2 + 2(a/c - 1) \right] \]

and

\[ B = \sqrt{2} \frac{M_p}{M_{pw}} (a/c - 1). \]

This upper bound, computed for the 8 WF 40 section, is shown in Table 7 and plotted in Fig. 21 as a function of \( \ell/c \). It is seen that the curve for the upper bound load \( P* \) lies parallel to and only slightly higher than that for the lower bound load \( \ell_2 \) corresponding to shear failure in the web.

F. On the relations between experiments and theory.

The example of the curve \( P_2\ell/M_p \) versus \( \ell/c \) in Fig. 21 clearly shows that the theory predicts a certain length ratio of the beam below which the load carrying capacity of the beam decreases abruptly, because of shear failure of the web. This critical length ratio is \( \ell_3/c \). In the following comparison we
therefore look first for the appearance of such a critical length ratio in the experiments, and, when observed, we compare the values of the experimental and the theoretical length ratios.

Secondly we want obviously to compare the experimental and theoretical values of the limit moment in each case. The theoretical values are given by equations (35), (37) (which furnish lower bounds) and (38) (which gives an upper bound, valid for short beams). In the following experimental data from three sources are considered. In the paper by Johnston, Yang and Beedle [13] shear failures in the web of I-beams are reported, but details concerning the length ratios and limit moment values are not given. Baker and Roderick [11] made an investigation of the influence of shear on the limit moment of a particular beam section and presented the results in the form of load-deflection curves from which the variation of limit moment with length ratio can be studied. Hendry [12] has discussed tests on a series of beams of different cross-sections and gives the results in a table where the limit moment was defined by deflection considerations. Adopting his definition of limit moment comparisons are made in this section between the experimental results and the theoretical predictions both as to critical length ratio and magnitude of limit moment.

In the eighth Progress Report from Lehigh University [13] two examples of beams are shown where the collapse of the beams was apparently caused by shear failure in the web. The first example is a 4 I 7.7 beam that is simply supported and loaded symmetrically by two concentrated forces. Each end of
the beam can therefore be represented by the cantilever problem discussed in this paper. The length ratio \( \ell/c \) in the test is approximately 2.5. In the theory equation (34) gives \( \ell_3/c = 2.5 \), as shown in Table 1.

In the second example an 8 WF 40 beam is loaded so that moments numerically equal but opposite in sense act at the ends of the half-beam, whose length ratio was approximately 16. Because of the symmetrical properties of the load each quarter of the span can be represented by the cantilever beam. The effective length ratio therefore is \( \ell/c = 8 \). For this case we compute \( \ell_3/c = 6.0 \).

The ratio between the collapse moment and the fully plastic moment is not stated for these two examples. The tests show that shear failure in the web is important for beams having the above cross-sections and the effective length ratio approximately \( \ell_3/c \).

Simply supported beams acted upon by two concentrated loads, symmetrically spaced, have been investigated and reported by Baker and Roderick [11]. The beam is a British H-section with dimensions 1\( \frac{1}{4} \)" x 1\( \frac{1}{4} \)". The distance between the loads was kept constant and the span was varied. Different effective length ratios of the corresponding cantilever beam problem were therefore obtained, varying from 9.4 (Test No. SR 21) to 2.2 (Test No. SR 11). The value of \( \ell_3/c \) is 6.7. The load-deflection curves are given. For length ratios greater than 3.9 the curves show a well defined "bend" where the slope of the curve rapidly decreases. The load at these "bends" is
taken as the collapse load in Table 8 and Fig. 22. The table and the figure show that in the experiment when the length ratio is smaller than 5.5 the collapse moment decreases rapidly with further decrease in length ratio. In the theory we have $l_3/c = 6.7$ below which $P_2M_p$ decreases rapidly.

In the paper by Hendry [12] the influence of shear forces in simply supported beams of various I-sections is discussed. In these tests a concentrated load acts on the midpoint of the beam. A series of tests is also included in which the influence of shear stresses in the beam of a portal is investigated. The value of $M/M_p$ is tabulated in the paper, where the moment $M$ is obtained from the load at which the deflection of the beam equals 1/50 of the span. These values are shown in Table 9 and Figs. 23 and 24 together with the theoretical values of $P_l/M_p$. Except for the portals and the beam 3b we see that $M/M_p$ is in the neighborhood of and is greater than $P_2l/M_p$. We notice moreover that for length ratios shorter than $l_3/c$ the value $M/M_p$ is considerably less than unity.

The portals Bl - B5 are made up of a 3" x 1" beam and of 1½" x 1" stanchions. The ratio of the fully plastic moment of the beam to that of the stanchion is approximately 7.3 to 1. The effective length ratio $l/c$ is therefore taken to be

$$\frac{7.3}{7.3 + 1} (l/c)^* = 0.88 (l/c)^*$$

where $(l/c)^*$ is the length ratio between the load and the end of the beam. The results in Table 9 show that the portals did not develop the moment $P_2l/M_p^*$. 
In [12] the large reduction of the limit moment is explained by the influence of the axial stresses in the columns.

In general we see that the theory and experimental tests agree rather well, with very few exceptions. For this comparison between the experiments and the theory we have used the results obtained by the second type of stress field. This is done because the length ratio is in these cases so short that the first type of stress field does not apply.

G. **Summary and conclusions.**

The two types of stress field in this paper each give an approximate lower bound for the true collapse load of the plastic bending of a cantilever I-beam. The approximations consist in both cases of a two-dimensional formulation of the problem while in the first type of stress field the further assumption is made that the \( \sigma_z \) stress is negligible. The velocity field, which is applied for short beams, gives results which are upper bounds to the true collapse load. The velocity field satisfies all the requirements of the appropriate theorem of limit analysis. The upper and lower bounds differ by small amounts for short beams. For long beams the lower bounds furnished by both types of stress field tend toward the result \( P_l/M_p = 1 \).

The ratio \( P_2\ell/M_p \), where \( P_2 \) is the lower bound on the collapse load defined in Section D, can be used to estimate the limit moment \( M_0 = P_2\ell \) in the relation to the fully plastic moment \( M_p \) which is appropriate for pure bending (or very long
cantilevers). We note that the curve of $P_2 l/M_p$ as function of $l/c$ has two distinct parts, as exemplified by the curves of Figs. 21 - 24. For $l/c \leq l_3/c$ it is a straight line through the origin with the slope $\frac{a/c + 1}{\sqrt{3} M_p/M_{pw}}$. For $l/c \geq l_3/c$ the value of $P_2 l/M_p$ increases asymptotically to the value $P_2 l/M_p = 1$.

The reduction of the limit moment at the length ratio $l_3/c$ may be as high as 30% for the beams mentioned in Table 1. The curves show that a much larger reduction of the limit moment takes place when the length ratio is less than $l_3/c$, but for beams having a length ratio greater than $l_3/c$ the fully plastic moment is generally reduced by a relatively small amount. Table 1 also shows that the value of $l_3/c$ in some cases is so short that in the practical problem the effects of the details of the end constraint or of the loading are very important. It is therefore suggested that beams should not be used whose length ratio is smaller than $l_3/c$ or which are so short that the end effects are important. For example, if $l/c < 6$ treatment as a "beam" in the conventional sense is hardly appropriate.

The influence of buckling, particularly of the web or the flange, (or both), is disregarded in this paper. This may be important for certain short beams, as indicated in [13], and should be further investigated.
Bibliography


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<th>Beam</th>
<th>See Ref</th>
<th>a</th>
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<th>b₁</th>
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<th>$l_1/c$</th>
<th>$l_2/c$</th>
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<th>$P_{2i}/M_p$ at $i_3/c$</th>
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Table 1. Beam dimensions and computed values
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**Table 2. The function u/c for**

**8 WF 40 section; l/c = 21.1**

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**Table 3. The function u/c in zone BC for**

**8 WF 40 section; l/c = 12.5356**
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<th>f</th>
<th>u/c</th>
<th>ϕ</th>
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<th>g/c√1+2sin²f</th>
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Table 4. The functions f, u/c, ϕ and g/c in zone DE for 8 WF 40 section; l/c = 12.5356
Table 5. The functions \( f, h, \varphi \) and \( g/c \) in zone FG for 8 WF 40 section: \( l/c = 10.53492 \)

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<th>( y )</th>
<th>( f )</th>
<th>( h )</th>
<th>( \varphi )</th>
<th>( g/c )</th>
<th>( g/c \sqrt{1+2\sin^2 f} )</th>
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Table 7. The values of \( P_1 l/M_p \), \( P_2 l/M_p \) and \( P^* l/M_p \) for 8 WF 40 section

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<td>( P_2 l/M_p )</td>
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Table 6. The functions \( f, u/c, \phi \) and \( g/c \) in zone DF for 8 WF 40 section; \( \frac{1}{c} = 10.53352 \)
Table 8. The tests by Baker and Roderick

<table>
<thead>
<tr>
<th>Test No.</th>
<th>l</th>
<th>l/c</th>
<th>2W</th>
<th>Wl/c</th>
<th>P_*l/M_p</th>
<th>P^*l/M_p</th>
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<td>4.25</td>
<td>9.44</td>
<td>2.05</td>
<td>9.68</td>
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<tr>
<td>20</td>
<td>3.75</td>
<td>8.33</td>
<td>2.30</td>
<td>9.58</td>
<td>0.904</td>
<td>&gt; 1</td>
</tr>
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<td>2.65</td>
<td>9.57</td>
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<td>3.05</td>
<td>9.32</td>
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<td>3.30</td>
<td>9.17</td>
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<td>0.773</td>
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<td>0.600</td>
</tr>
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<td>Beam No.</td>
<td>Section</td>
<td>l</td>
<td>l/c</td>
<td>( \frac{P_2 l}{M_p} )</td>
<td>( \frac{P_l l}{M_p} )</td>
<td>M/M_p</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------</td>
<td>-----</td>
<td>-----</td>
<td>--------------------------</td>
<td>--------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>1a</td>
<td>4&quot; x 3&quot; R.S.J.</td>
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<td>9.1</td>
<td>0.973</td>
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<td>0.95</td>
</tr>
<tr>
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<td>12</td>
<td>7.3</td>
<td>0.953</td>
<td>&gt; 1</td>
<td>1.00</td>
</tr>
<tr>
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<td>5.5</td>
<td>0.909</td>
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<td>1.00</td>
</tr>
<tr>
<td>d</td>
<td>( \ell_3/c = 4.5 )</td>
<td>6</td>
<td>3.6</td>
<td>0.684</td>
<td>0.719</td>
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<td>e</td>
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<tr>
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<td>7.2</td>
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<td>0.96</td>
</tr>
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<td>1.00</td>
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<tr>
<td>d</td>
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<tr>
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<td>0.98</td>
<td>0.61</td>
</tr>
<tr>
<td>g</td>
<td>Bl - B5</td>
<td>4</td>
<td>2.8</td>
<td>0.814</td>
<td>0.98</td>
<td>0.76</td>
</tr>
<tr>
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<td>1.00</td>
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<tr>
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<td>3</td>
<td>0.705</td>
<td>0.744</td>
<td>0.76</td>
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Table 9. The tests by Hendry
Fig. 6 The shape of the plastic regions for \( l \geq l_1 \) and \( l \geq l_0 \)

Fig. 7 The shape of the plastic regions for \( l_1 \geq l \geq l_2 \) and \( l \geq l_0 \)

Fig. 8 The shape of the plastic regions for \( l_2 \geq l \geq l_0 \)
Fig. 9. The ratio $\frac{P_1l}{M_p}$ of the first type of stress field
Fig. 10  The stresses at a cross-section in DE

Fig. 11  The stresses at a cross-section in FG
Fig. 12. The plastic region for 8WF4O section, \( \frac{r}{c} = 2.1 \)
Fig. 13. The function $f(x)$ for 8WF40 section; $U_c = 12.5356$
Fig. 14. The plastic regions for 8WF40 section; $l/c = 12.5356$
Figs. 15 and 16. The function $f(x)$ and the plastic regions for 8W-40 section, $\mu c = 0.53492$
Fig. 17. Shear failure in the web

Fig. 18. The second type of stress field

Fig. 19. The second type of stress field

Fig. 20. The kinematically admissible velocity field
Fig. 21. The theoretical values of the limit load for 8WF40 section.
Fig. 22. The beam tests by Baker and Roderick [11]

Fig. 24. The portal tests by Hendry [12]
Fig. 23. The beam tests by Hendry [12]