PROBABILITY DISTRIBUTIONS RELATED TO RANDOM TRANSFORMATIONS
OF A FINITE SET

BY

R. RUBIN AND R. SITGREAVES

TECHNICAL REPORT NO. 19A (final)

JANUARY 22, 1954

This work was sponsored by the Army, Navy, and Air
Force through the Joint Services Advisory Committee
for Research Groups in Applied Mathematics and
Statistics by Contract No. N6onr 25140 (MR-342-022)

APPLIED MATHEMATICS AND STATISTICS LABORATORY
STANFORD UNIVERSITY
STANFORD, CALIFORNIA
Introduction. Let \( X \) be a set of \( n \) elements, and let \( J \) be the set of all transformations of \( X \) into \( X \). In the definitions, the notation \( T^k \), where \( T \in J \), has its usual meaning, with \( k \) a positive integer or zero. For \( k = 0 \), \( T^0 x = x \).

For a given \( x \in X \) and \( T \in J \), the set of all elements \( y \in X \) such that
\[
T^j x = T^k y
\]
for at least one pair of numbers \( j, k \), is called the structure in \( T \) containing \( x \), and is denoted by \( S_T(x) \). Trivially, \( x \in S_T(x) \) since \( T^0 x = T^j x \) for all \( j \). The number of elements in \( S_T(x) \) is the size of the structure containing \( x \).

An element \( y \in X \) is called a cyclical element in \( T \) if \( T^m y = y \) for some \( m > 0 \). A cyclical element \( y \) belongs to a cycle of length \( k \), if
\[
T^j y / y, \quad 0 < j < k
\]
and
\[
T^k y = y.
\]
It follows that each element
\[
x = T^j y, \quad 0 < j < k
\]
is also a cyclical element in \( T \), belonging to the same cycle as \( y \).
since for any $m$

$$T^m z = T^m (T^i y) = T^{m+i} y = T^j (T^m y)$$

so that

$$T^m z = T^j (T^m y) \neq T^i y = z \text{ for } 0 < m < k$$

and

$$T^k z = T^j (T^k y) = T^i y = z$$

Every structure $S_T(x)$ contains a non-empty subset of cyclical elements, say $K_T(x)$. Clearly, the elements of $K_T(x)$ represent a single cycle, since for any cyclical elements $y, z$ with $y \in S_T(x)$ and $z \in S_T(x)$, there exist numbers $j, k$ such that

$$T^j y = T^k z$$

An element $y \in S_T(x)$ is said to be a predecessor of $x$ if there exists a $k \geq 0$ such that $T^k y = x$. An element $y$ is said to be a successor of $x$ if there exists a number $k \geq 0$ such that $T^k x = y$.

The set of all successors of $x$ is called the six-length of $x$. It will be noted that $x$ is both a successor and a predecessor of itself. If $x \in K_T(x)$, every element in $S_T(x)$ is a predecessor of $x$, while the set $K_T(x)$ is the six-length of $x$.

Consider the following functions defined on $\mathbb{N}$:

- $m =$ number of structures in $T$
- $c =$ size of the structure containing $x$
- $s =$ number of elements in the six-length of $x$
- $p =$ number of predecessors of $x$.
We assume that elements are selected at random from \( \mathcal{X} \), with each pair \((x, T)\) having probability \( \frac{1}{n^{n-1}} \) of being chosen. Exact probability distributions are derived for the functions \( m, c, s, \) and \( p \), and asymptotic approximations are given for the distributions of \( c, s, \) and \( p \) as \( n \) becomes large.

2. **Graphical representation of the set \( \mathcal{J} \)**. Let \( \mathcal{J}^* \) be the subset of \( \mathcal{J} \) consisting of all transformations of \( \mathcal{X} \) onto \( \mathcal{X} \), i.e., \( \mathcal{J}^* \) is the set of all permutations of the elements of \( \mathcal{X} \). Then, two elements of \( \mathcal{J} \), say, \( T_i \) and \( T_j \), are said to belong to the same structure class, if for at least one \( T^* \in \mathcal{J}^* \)

\[(2.1) \quad T_i - T_j - T^* \]

It is clear that the maximum number of elements of \( \mathcal{J} \) which can belong to the same structure class is \( n! \).

A graphical presentation of the various structure classes when \( n = 5 \) is given in Figure 1. The number of elements of \( \mathcal{J} \) belonging to each class is also given. The classes are grouped according to the value of \( m \), the number of structures in the transformation, and \( N^* \), the total number of cyclical elements.

3. **Auxiliary distributions**. Let \( D \) be some subset of \( \mathcal{X} \) containing \( d \) elements, and let \( \mathcal{J}_D \) be the subset of \( \mathcal{J} \) containing all transformations \( T \) for which \( D \supseteq TD \). For any pair \((x, T)\) let \( \ell \) denote the length of the cycle in \( S_T(x) \). Then

\[(3.1) \quad P \left\{ x_T(x) \subseteq X - D, \ s = k, \ \ell = j (j \leq k) \mid D, T \in \mathcal{J}_D \right\} = P \left\{ x \notin D, T \notin X \text{ and } T \notin D, T^2x \notin X \text{ or } T^x \right\} \]
and $T^2 x \not\in D$; \ldots; $T^{k-1} x \not\in x$, $Tx$, \ldots, or $T^{k-2} x$,
and $T^{k-1} x \not\in D$; and $T^k x = T^{k-1} x \mid D$

\[
\frac{n-d}{n} \cdot \frac{n-d-1}{n} \cdots \frac{n-d-k+1}{n} \cdot \frac{1}{n}
\]

\[
= \frac{(n-d)!}{(n-d-k)! \cdot n^{k+1}}
\]

Or

(3.2) \quad P \left\{ x \not\in K_T(x), \ K_T(x) \subseteq X \implies D \quad | \quad \lambda = j \mid D, \ T \in J_D \right\}

\[
= \sum_{k=j+1}^{n-d} \frac{(n-d)!}{(n-d-k)! \cdot n^{k+1}}
\]

for $j = 1, 2, \ldots, n-d$.

For any $T \in J_D$, let $N_D$ be the number of cyclical elements in $X \subseteq D$. Now, for any $T \in J_D$ and any $x \not\in D \cup K_T(x)$, either $K_m(x) \subseteq D$

or $K_T(x) \subseteq X \implies D$. Hence

(3.3) \quad P \left\{ x \not\in K_T(x), \ K_T(x) \subseteq X \implies D \quad | \quad \lambda = j \mid D, \ T \in J_D \quad N_D = q \right\}

\[
= P \left\{ x \not\in D \cup K_T(x) \mid D, \ K_T(x) \quad N_D = q \right\}
\]

\[
= P \left\{ K_T(x) \subseteq X \implies D \mid x \not\in D \cup K_T(x), \ D, \ N_D = q \right\}
\]

\[
= P \left\{ \lambda = j \mid x \not\in D \cup K_T(x), \ K_T(x) \subseteq X \implies D, \ D, \ N_D = q \right\}
\]

\[
= \frac{n-d-c}{n} \cdot \frac{q}{d+q} \cdot \frac{1}{q}
\]

\[
= \frac{n-d-c}{n(d+q)}
\]

Or
Comparing (3.2) and (3.4) for \( k = j \) and \( k = j+1 \), we obtain

\[
(3.5) \quad P \left\{ x \not\in K_{k}(x), \ K_{k}(x) \subseteq X \rightarrow D, \ k = j \mid D, \ T \in J_{D} \right\} = \frac{n-d-j}{n-d+1} \cdot \frac{n-d}{(n-d-j) \cdot n^{j+1}} \quad \text{for} \quad j = 1, 2, \ldots, n-d.
\]

It follows that

\[
(3.6) \quad P \left\{ N_{D} = j \mid D, \ T \in J_{D} \right\} = \frac{(n-d) j (d+j)}{(n-d-j) \cdot n^{j+1}} \quad \text{for} \quad j = 1, 2, \ldots, n-d.
\]

It remains to evaluate \( P \left\{ N_{D} = 0 \mid D, \ T \in J_{D} \right\} \). For this purpose, consider the function \( g \) of a real variable \( t \), where

\[
g(t) = \frac{n-d}{\sum_{j=0}^{d} \frac{(n-d) j (d+j) t^{d+j}}{(n-d-j) \cdot n^{j+1}}}
\]

Let

\[
f(t) = \frac{n-d}{\sum_{j=0}^{d} \frac{(n-d) j t^{j}}{(n-d-j) \cdot n^{j+1}}}
\]

Then

\[
g(t) = t^{d+1} f'(t) + d t^{d} f(t).
\]

But
\[
f'(t) = \sum_{j=0}^{n-d} \frac{(n-d)! (t^j - t)}{(n-d-j)! n^j !}
\]

so that

\[
g(t) = n(t^d - t^{d-1}) f(t) + \frac{1}{t^2}
\]

and \(g(1) = 1\). But

\[
g(1) = \sum_{j=0}^{n-d} \frac{(n-d)! (d+j)}{(n-d-j)! n^j !}
\]

and

\[
\mathbb{P}\left\{ N_D = 0 \mid D \in J_D \right\} = 1 - \sum_{j=0}^{n-d} \frac{(n-d)! (d+j)}{(n-d-j)! n^j !} + \frac{d}{n}
\]

so that

\[
(3.7) \quad \mathbb{P}\left\{ N_D = 0 \mid D \in J_D \right\} = \frac{d}{n}
\]

If \(D = \emptyset\) where \(\emptyset\) is the empty set, \(d = 0\) and \(J_\emptyset = J\). For any \(T \in J\), \(N_T\) is thus the total number of cyclical elements in \(X\).

From (3.6) and (3.7) we obtain

\[
(3.8) \quad \mathbb{P}\left\{ N_T = j \right\} = \frac{(n-1)! j}{(n-j)! n^j} \quad j = 1, 2, \ldots, n
\]
1. **Distribution of the number of structures.** We can write

\[(4.1) \quad P\{m=i\} = \sum_{j=1}^{n} P\{m=i, N_j=j\} = \sum_{j=1}^{n} P\{m=i | N_j=j\} P\{N_j=j\}\]

But

\[P\{m-i | N_j=j\} = \text{Prob. of } i \text{ cycles in a permutation of } j \text{ elements} = \alpha(i,j), \text{ say}\]

we have

\[
\begin{align*}
\alpha(1,1) &= 1 \\
\alpha(1,2) &= \frac{1}{2} \\
\alpha(2,2) &= \frac{1}{2} \\
\alpha(2,3) &= \frac{1}{2} \\
\alpha(3,3) &= \frac{1}{8}
\end{align*}
\]

In general

\[\alpha(i,j) = \frac{j-1}{j} \alpha(i,j-1) + \frac{1}{j} \alpha(i-1,j-1)\]

for \(i \leq j\). Clearly \(\alpha(i,j) = 0\) for \(i > j\). It will be noted that for each \(j\), \(\alpha(1,j) = \frac{1}{j}\) and \(\alpha(j,j) = \frac{1}{j}\). Also, we have

\[\alpha(i,j) = \text{coef. of } t^i \text{ in } \frac{\Gamma(t+j)}{\Gamma(t)j!}\]

The distribution of cycles in permutations of a finite number of elements has also been considered by Gontcharoff [1].

It follows that

\[(4.2) \quad P\{m=i\} = \sum_{j=1}^{n} \frac{(n-1)!1!}{(n-j)!j!} \alpha(i,j)\]

Values of \(\alpha(i,j)\) for \(i, j = 1, 2, \ldots, 25, 1 \leq j\), are given in Table 1.
5. Distribution of structure size. Let $X_j$ be a subset of $X$ containing $j$ elements. Also, let $J(j)$ be the set of all transformations of $X_j$ into itself, and for any $T \in J(j)$, let $m(T)$ denote the number of structures in $T$.

For any pair $(x, T)$ selected at random from $X \times J$, the size of the structure containing $x$ is the number of elements in $S_r(x)$. Then the probability that a picked structure has size $j$ is given by

\[
P\{X_j \text{ forms the picked structure} \} = \binom{n}{j} P\{X_1 \in X_j\} \cdot P\{TX_j \subseteq X_j\} \cdot P\{T(X_1) = X_1\} \cdot P\{m(T) = j\}
\]

From (4.2) we have

\[
P\{m(T) = 1\} = \sum_{k=1}^{j} \frac{(j-1)^k}{(j-k)^8 j^k} \alpha_{(1, l)}
\]

It follows that

\[
P\{c = j\} = \sum_{k=1}^{j} \frac{(j-1)^k}{(n-j)^8 (j-k)^8 j^k} \cdot \frac{1}{(n-j)^8 j^k}
\]

6. Distribution of six-lengths. We obtain the joint distribution of six-length and cycle length directly from (3.1) by letting $D = \emptyset$. Thus
It follows that
\[
(6.2) \quad P \{ s = k \} = \frac{(n-1)^2 k}{(n-k)^2 n^2}
\]
and
\[
(6.3) \quad P \{ \ell = j \} = \sum_{k=j}^{n} \frac{(n-1)^2}{(n-k)^2 n^2}
\]
The expected six-length is given by
\[
(6.4) \quad E(s) = \sum_{k=1}^{n} \frac{(n-1)^2 k^2}{(n-k)^2 n^2}
\]
Writing \( k^2 = n^2 - n(n-k) = (n-1)(n-k) + (n-k) \), this becomes
\[
(6.5) \quad E(s) = n \left( \sum_{k=1}^{n} \frac{(n-1)^2}{(n-k)^2 n^2} \right) - \sum_{k=1}^{n-1} \frac{(n-1)^2}{(n-k)^2 n^2} - \sum_{k=1}^{n-1} \frac{(n-1)^2}{(n-k)^2 n^2}
\]
\[
= n \left( \sum_{k=1}^{n} \frac{(n-1)^2}{(n-k)^2 n^2} \right) + \sum_{k=1}^{n-1} \frac{(n-1)^2}{(n-k)^2 n^2} - \sum_{k=1}^{n-1} \frac{(n-1)^2}{(n-k)^2 n^2}
\]
\[
= n \left( \sum_{k=1}^{n} \frac{(n-1)^2}{(n-k)^2 n^2} \right) + \sum_{k=1}^{n-1} \frac{(n-1)^2}{(n-k)^2 n^2}
\]
\[
= n \left( \sum_{k=1}^{n} \frac{(n-1)^2}{(n-k)^2 n^2} \right)
\]
\[
= \sum_{k=1}^{n} \frac{(n-1)^2}{(n-k)^2 n^2}
\]
It is of interest to note that

\[(6.7) \quad E(\frac{1}{n}) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)! n^k} = \frac{1}{n} E(n)\]

7. **Distribution of predecessors.** Let \(x\) be a given element in \(X\), and now let \(X_j = x \cup X_{j-1}^*\) where \(X_{j-1}^*\) is a subset of \(X-x\) containing \((j-1)\) elements. As before, let \(J(j)\) be the set of all transformations of \(X_j\) into itself, and let \(J(j)_X\) be the subset of \(J(j)\) consisting of all transformations \(T\) for which \(T(x) = x\). For any \(T \in J(j)_X\), let \(N^*_x(j)\) be the number of cyclical elements in \(X_j - x\), i.e., in \(X_{j-1}^*\). Then the probability that \(x\) has \(j\) predecessors, where \(x\) is counted as a predecessor of itself, is given by

\[(7.1) \quad P\{p = j|x\} = \frac{\text{Number of ways in which } X_{j-1}^* \text{ can be chosen}}{P\{T(X_j - x) \subset X - X_j\} \cdot P\{T(X_j - x) \subset X - X_j\}^j} \cdot \frac{N^*_x(j)}{N^*_x(j)} \]

\[= \frac{(n-1)!}{j!} \left(\frac{1}{n}\right)^{j-1} \left(\frac{n-1}{n}\right)^{n-j} \cdot \frac{1}{j} \]

\[= \frac{(n-1)! \cdot j^{j-2} \cdot (n-j)^{n-j}}{(n-j)! \cdot (j-1)! \cdot n^{n-1}}\]

It follows that

\[(7.2) \quad P\{p = j\} = \sum_{i=1}^{n} \frac{1}{n} \left(\frac{(n-1)! \cdot j^{j-2} (n-j)^{n-j}}{(n-j)! \cdot (j-1)! \cdot n^{n-1}}\right)\]

\[= \frac{(n-1)! \cdot j^{j-2} (n-j)^{n-j}}{(n-j)! \cdot (j-1)! \cdot n^{n-1}}\]
8. Distribution of \( m, c, s, \) and \( \eta \) when \( n = 5 \). The probability distributions of \( m, c, s, \) and \( \eta \) when \( n = 5 \) are as follows:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( P{n = k} )</th>
<th>( P{c = k} )</th>
<th>( P{s = k} )</th>
<th>( P{\eta = k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1569/3125</td>
<td>256/3125</td>
<td>125/625</td>
<td>256/625</td>
</tr>
<tr>
<td>2</td>
<td>1220/3125</td>
<td>321/3125</td>
<td>200/625</td>
<td>108/625</td>
</tr>
<tr>
<td>3</td>
<td>305/3125</td>
<td>108/3125</td>
<td>180/625</td>
<td>72/625</td>
</tr>
<tr>
<td>( \infty )</td>
<td>30/3125</td>
<td>560/3125</td>
<td>96/625</td>
<td>64/625</td>
</tr>
<tr>
<td>5</td>
<td>1/3125</td>
<td>1569/3125</td>
<td>24/625</td>
<td>125/625</td>
</tr>
</tbody>
</table>

These distributions are shown graphically in Figure 2.

9. Asymptotic expansion of \( \sum_{j=1}^{k} \frac{(k-1)^{j}}{(k-j)!} \). An asymptotic expansion for the quantity \( \sum_{j=1}^{k} \frac{(k-1)^{j}}{(k-j)!} \) which appears in the probability distribution of structure size as well as the expression for the expected six-length, can be obtained as follows: We have

\[
(9.1) \quad \sum_{j=1}^{k} \frac{(k-1)^{j}}{(k-j)!} \cdot \frac{(k-1)^{j}}{(k-j-1)!} \cdot \frac{1}{k} = \frac{1}{k} \int_{0}^{\infty} e^{-x} \left( \sum_{j=0}^{k-1} \frac{(k-1)^{j}}{(k-1-j)!} \cdot \left( \frac{x}{k} \right)^{j} \right) dx
\]

\[
= \frac{1}{k} \int_{0}^{\infty} e^{-x} (1 + \frac{x}{k})^{k-1} dx
\]
Let \( x = k(e^y - 1) \), \( dx = ke^y \, dy \). It follows that

\[
\frac{1}{k} \int_0^\infty e^{-x} (1 + \frac{x}{k})^{k-1} \, dx = \int_0^\infty e^{-k(e^y-1-y)} \, dy
\]

\[
= \int_0^\infty e^{-\frac{m}{y^{3/2}}} y^{3/2} \, dy .
\]

Now let \( y = (2u/k)^{3/2} \), \( dy = (3/(2ku)^{3/2}) \, du \)

so that

\[
\int_0^\infty e^{-k(y^{3/2})} \, dy = \int_0^\infty \frac{1}{(2ku)^{3/2}} e^{-u} \, du
\]

\[
= \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u} u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j} \, du
\]

\[
= \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u} u \sum_{j=0}^{\infty} \frac{b_j(u)}{k^j} \, du
\]

where

\[
a_0(u) = 1 \\
a_1(u) = u^3/9 - u^2/6 \\
a_2(u) = u^6/486 - u^5/34 + 13u^4/360 - u^3/90 \\
a_3(u) = u^9/65,610 - u^8/2,916 + 23u^7/9,720 - 37u^6/6,480 \\
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\]

\[
b_0(u) = 1 \\
b_1(u) = u^3/27 - u^2/6 + u/10 \\
b_2(u) = u^6/2,430 - u^5/162 + u^4/40 - u^3/36 + u^2/210 \\
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\]
Consider the integral
\[ \int_0^\infty e^{-k(e^{-y} - 1 - y)} \, dy = \int_0^\infty e^{-k\sum_{j=2}^\infty (-1)^j j^{1/j}} \, dy \]
\[ \quad = -u + \frac{\sqrt{2u}}{3k^{3/2}} - \frac{u^2}{6k} + \frac{\sqrt{2u}^5/2}{30k^{3/2}} \ldots \]
\[ \quad = \int_0^\infty \frac{1}{(2ku)^{1/2}} \, e^{-u} \, du \]
\[ \quad = \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u} u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j} \, du \]
\[ \quad + \frac{1}{2k} \int_0^\infty e^{-u} u \sum_{j=0}^{\infty} \frac{b_j(u)}{k^j} \, du \ldots \]
\[ \quad \text{Or} \]
\[ \quad \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u} u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j} \, du \]
\[ \quad = \frac{1}{2} \int_0^\infty \left( e^{-k(e^{-y} - 1 - y)} + e^{-k(e^{-y} - 1 + y)} \right) \, dy \]
\[ \quad = \frac{1}{2} \int_0^\infty e^{-k(e^{-y} - 1 + y)} \, dy \]
\[ \quad \text{Let } y = \log t \quad dy = \frac{dt}{t} \]
\[ \quad \text{Or} \]
\[ \quad \frac{1}{(2k)^{1/2}} \int_0^\infty e^{-u} u^{-1/2} \sum_{j=0}^{\infty} \frac{a_j(u)}{k^j} \, du = \frac{1}{2} e^k \int_0^\infty e^{-kt} t^{k-1} \, dt \]
\[ \quad = \frac{1}{2} \left( e/k \right)^k \Gamma(k) \ldots \]
Thus we obtain

\( (9.3) \quad \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)! k^j} \sim \frac{e^k f(k)}{2k^k} - \frac{1}{3k} - \frac{1}{135k^2} + \frac{3}{2.835k^3} \ldots \ldots \ldots \)

10. Asymptotic approximation for the distribution of structure size. We have (cf. 5.2)

\( (10.1) \quad P\{ s = k \} = \frac{(n-1)!}{(n-k)! (k-1)!} \frac{k^k (n-k)^{n-k}}{n^n} \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)! j^j} . \)

The mode of the distribution is at \( k = n \). We have

\( (10.2) \quad P\{ s = n \} = \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)! j^j} . \)

An approximation for this expression is given in the preceding section.

Except at the tails, i.e., if neither \( k \) nor \( n-k \) is small, we have

\( (10.3) \quad P\{ s = k \} \sim \frac{e^{-n} n^{-1/2} (2n)^{1/2} k^k (n-k)^{n-k}}{e^{-(n-k)-k} (n-k) k^{k-1/2} (2n) n^n} \sqrt{\frac{n}{2\pi}}

= \frac{1}{2n^{1/2} (n-k)^{1/2}} . \)

The asymptotic density of \( x = \frac{s}{n} \) is given by

\[ p(x) = \frac{1}{2(1-x)^{1/2}} \quad 0 \leq x \leq 1. \]

11. Asymptotic approximation for the distribution of size-lengths, we have (cf. 6.2)

\( (11.1) \quad P\{ s = k \} = \frac{(n-1)! k}{(n-k)! n^k} . \)
The mode of the distribution is at $k = \sqrt{n}$. Let $k = \sqrt{n} x$. Then, as $n$ becomes large

\[
(11.2) \quad P\{s = \sqrt{n} x\} \sim \frac{e^{-n(n - x^2)} (2\pi)^{1/2} n^{-1/2} x}{e^{-n(n - x^2)} (n - x)^n x + 1/2(2\pi)^{1/2} n^{-1/2} x}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

\[
\quad \sim \frac{x}{e^{n} x n^{1/2} (1 - \frac{x}{n}) n^{-1/2} (1 - \frac{x}{n})}
\]

The asymptotic density of $\frac{x}{\sqrt{n}}$ is thus

\[
(11.3) \quad P(x) = x e^{-(1/2)x^2}
\]

12. Asymptotic approximation for the distribution of predecessors. The distribution of the number of predecessors (cf. 7.2) is

\[
(12.1) \quad P\{p = k\} = \frac{(n-1)_k (n-k)^{n-k}}{(n-k)! (n-1)! n^{n-1}}
\]

This is a U-shaped distribution with an antimode at $(3/4)n$. The probability that $p = n$ is given by

\[
P\{p = n\} = \frac{1}{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]
As \( n \) becomes large and \( k \) remains small relative to \( n \), we have, using Stirling's approximations for \((n-1)!\) and \((n-k)!\)

\[
(12.2) \quad \lim_{n \to \infty} P\{p = k\} = \frac{k^{k-2} e^{-k}}{(k-1)!}
\]

As \( k \) increases, but is still small relative to \( n \), we have

\[
(12.3) \quad P\{p = k\} \sim \frac{1}{\sqrt{2\pi k^{3/2}}}
\]

Values of the limiting probability that \( p > k \) have been computed for some selected values of \( k \) and \( n \). These values follow

\[
P\{p > k\} \quad (\text{in percent})
\]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( n = 5 )</th>
<th>( n = \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>59.04</td>
<td>63.21</td>
</tr>
<tr>
<td>2</td>
<td>41.76</td>
<td>49.68</td>
</tr>
<tr>
<td>3</td>
<td>30.24</td>
<td>42.21</td>
</tr>
<tr>
<td>4</td>
<td>20.00</td>
<td>37.33</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>33.82</td>
</tr>
</tbody>
</table>

\[
\text{n = } \infty
\]

<p>| 10  | 24.55 |
| 25  | 15.782 |
| 100 | 7.957 |
| 250 | 5.041 |
| 1,000 | 2.522 |
| 2,500 | 1.5956 |
| 10,000 | 0.7979 |
| 25,000 | 0.5046 |</p>
<table>
<thead>
<tr>
<th>k</th>
<th>n = 10^8</th>
<th>n = 10^9</th>
<th>n = 10^{10}</th>
<th>n = \infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^5</td>
<td>.2522</td>
<td>.2523</td>
<td>.2523</td>
<td>.2523</td>
</tr>
<tr>
<td>2.5 \times 10^5</td>
<td>.15938</td>
<td>.15956</td>
<td>.159578</td>
<td>.15958</td>
</tr>
<tr>
<td>10^6</td>
<td>.07939</td>
<td>.07975</td>
<td>.079786</td>
<td>.07979</td>
</tr>
<tr>
<td>2.5 \times 10^6</td>
<td>.04983</td>
<td>.05040</td>
<td>.050454</td>
<td>.05046</td>
</tr>
<tr>
<td>10^7</td>
<td>.02394</td>
<td>.02510</td>
<td>.025217</td>
<td>.02523</td>
</tr>
</tbody>
</table>
13. **Case of each element having 0 or k predecessors.** Let \( J \) be the set of transformations of \( I \) into \( I \) such that each element \( x \in X \) has either no immediate predecessors or exactly \( k \) immediate predecessors, where \( k \) is a given number. For a given \( T \in J \), let \( A_0 \) be the set of elements with no immediate predecessors, and \( A_k = I - A_0 \) the set with \( k \) immediate predecessors. Also, let \( a_0 \) and \( a_k = n - a_0 \) be the number of elements in \( A_0 \) and \( A_k \), respectively. Since \( TX = A_k \), we must have \( n = ka_k \); that is, for the set \( J \) to be non-empty, \( n \) must be a multiple of \( k \), say, \( rk \), and \( a_k = r \) for all \( T \in J \).

The total number of transformations in \( J \) is \( Q(r,k) \), say, where

\[
Q(r,k) = \frac{r!(r(k-1))!k!}{(rk)!(rk)!rk}
\]

Clearly, the probability that any pair \((x,T)\) will be chosen is

\[
\frac{r!(r(k-1))!k!}{(rk)!(rk)!rk}
\]

A procedure similar to that used in the general case is again useful in deriving exact distributions for the functions \( m, o, s, \) and \( p \). Let \( D \) be a given subset of \( X \) containing \( d \) elements, and let \( J_D \) be the subset of \( J \) containing all transformations \( T \) for which \( D \supseteq TD \). Let \( J_D^{(TD)} \) be the subset of \( J_D \) for which \( TD \) is the specified transformation on \( D \). For any \( T \in J_D^{(TD)} \), let \( A_k^{(TD)} \) be the subset of \( D \) specified by \( TD \) as belonging to \( A_k \); i.e.,

\( TD = A_k^{(TD)} \) for all \( T \in J_D^{(TD)} \). Suppose that for a given \( TD \), \( A_k^{(TD)} \) contains \( d \) elements, and suppose we know \( D \) and \( TD \) only. Clearly, \( d \leq k \psi \) and \( d \leq k \psi \), so that \( k \psi - d \) predecessors for the elements of \( A_k^{(TD)} \) remain to be chosen from \( X - D \). Moreover, if \( d \leq r \), any of the \((d - \psi)\) elements of \( D - A_k^{(TD)} \)
for which no predecessor is specified by TD, may also be in \( A_k \), so that there are altogether

\[(13.3) \quad kV - d + k(d - \nu) = (k-1)d\]

possible immediate predecessors for elements of \( D \) in \( X - D \). In case \( d > r \), as many as \( (r - \nu) \) of the \( (d - \nu) \) elements of \( D - A_k^{(TD)} \) may also be in \( A_k \), so that there are altogether

\[(13.4) \quad kV - d + k(r - \nu) = rk - d\]

possible immediate predecessors for elements of \( D \) in \( X - D \). In either case, the number of possible immediate predecessors for elements of \( D \) in \( X - D \) depends only on \( d \), and not on the particular TD selected. We can write

\[(13.5) \quad d = k(r - a) - b \quad \text{or} \quad kr - d = ka + b\]

where \( b \) is given by \( (13.3) \) if \( d \leq r \) and by \( (13.4) \) if \( d > r \).

Now

\[(13.6) \quad P\left\{ x \in A_o, k_T(x) \subset X - D, s = j, \ell = q, q < j \mid D, T \in \mathcal{J}_D \right\}
= P\left\{ x \in A_o, x \notin D; T x \notin x \text{ and } T x \notin D; T^2 x \notin x \text{ or } T x, \text{ and } T^2 x \notin D; \ldots; T^{j-1} x \notin x, T x, \ldots, \text{ or } T^{j-1} x, \text{ and } T^j x \notin D; \text{ and } T^j x = T^j x \mid D, T \in \mathcal{J}_D \right\}
= \frac{(k-1)s!b \cdot k(a-1) \cdot k(a-2) \cdot \ldots \cdot k(a-b-1) \cdot (k-1)}{ka + b \cdot ka + b - 1 \cdot ka + b - 2 \cdot \ldots \cdot ka + b - j + 1 \cdot ka + b}
= \frac{((k-1)a^b)(k-1)k^j - a!(ka + b - 1)!}{(ka + b)!(ka + b)!(a - j)!}.

Also

\[(13.7) \quad P\left\{ x \in A_k, k_T(x) \subset X - D, s = j, \ell = q, q < j \mid D, T \in \mathcal{J}_D \right\}
= \frac{(k-1)k^j - a!}{(ka + b)!(ka + b)!(a - j)!}.

and
\[(13.8) \quad \Pr\{x \in \mathcal{L}, \mathbb{K}_T(x) \subset \mathcal{X}, a = j, \ell = j|D, T \in \mathcal{J}_D\} - \frac{j!a!(ka+b-1)!}{(ka+b)!((a-j)!)!} - \Pr\{\mathbb{K}_T(x) \subset \mathcal{X}, a = j, \ell = j|D, T \in \mathcal{J}_D\} \quad 1 \leq j \leq a\]

so that

\[(13.9) \quad \Pr\{\mathbb{K}_T(x) \subset \mathcal{X}, a = j, \ell = q, q < j|D, T \in \mathcal{J}_D\} - \frac{(k-1)j!a!(ka+b-j+1)!}{(ka+b)!((a-j+1)!)!} \quad 2 \leq j \leq a+1\]

and

\[(13.10) \quad \Pr\{x \notin \mathbb{K}_T(x), \mathbb{K}_T(x) \subset \mathcal{X}, \ell = q|D, T \in \mathcal{J}_D\} - \Pr\{x \notin \mathbb{K}_T(x), \mathbb{K}_T(x) \subset \mathcal{X}, \ell = q|D, T \in \mathcal{J}_D\} \quad 2 \leq j \leq a+1\]

As before, for any \(T \in \mathcal{J}_D\), let \(N_D\) be the number of cyclical elements in \(\mathcal{X}\). Then, we have

\[(13.11) \quad \Pr\{x \notin \mathbb{K}_T(x), \mathbb{K}_T(x) \subset \mathcal{X}, \ell = q|D, T \in \mathcal{J}_D, N_D = j\} - \Pr\{x \notin \mathcal{D} \cup \mathbb{K}_T(x)|D, T \in \mathcal{J}_D, N_D = j\}

- \Pr\{\mathbb{K}_T(x) \subset \mathcal{X}|x \notin \mathcal{D} \cup \mathbb{K}_T(x), D, T \in \mathcal{J}_D, N_D = j\}

- \Pr\{\ell = j|x \notin \mathcal{D} \cup \mathbb{K}_T(x), \mathbb{K}_T(x) \subset \mathcal{X}, D, T \in \mathcal{J}_D, N_D = j\} = \frac{(ka+b-1)}{ka+b} \cdot \frac{(k-1)!}{(k-1)j!} \cdot \frac{1}{j} - \frac{(k-1)(ka+b-1)}{(ka+b)((k-1)j+b)}\]

Or

\[(13.12) \quad \Pr\{x \notin \mathbb{K}_T(x), \mathbb{K}_T(x) \subset \mathcal{X}, \ell = q|D, T \in \mathcal{J}_D\} = \frac{1}{\sum_{j=q}^{a} \frac{(k-1)(ka+b-1)}{(ka+b)((k-1)j+b)} \Pr\{N_D = j|D, T \in \mathcal{J}_D\}}\]

Comparing (13.10) and (13.12) for \(\ell = q\) and \(\ell = q+1\), we obtain
(13.13) \[ P\{x \notin E(x), E(x) \subseteq X-D, \ell = q \mid D, T \in J_D \} \]

\[ = P\{x \notin E(x), E(x) \subseteq X-D, \ell = q+1 \mid D, T \in J_D \} \]

\[ = \frac{(k-1)(a+b-q)!}{(ka+b)((k-1)q+b)!} P\{E = q \mid D, T \in J_D \} \]

\[ = \frac{(k-1)q!(a+b-q)!}{(ka+b)(ka+b)!} \frac{1}{(a-q)!} \]

Or

(13.14) \[ P\{E = q \mid D, T \in J_D \} = \frac{(k-1)q!(a+b-q)!}{(ka+b)!} \frac{1}{(a-q)!} \]

for \(1 \leq q \leq n\). Since

(13.15) \[ \sum_{q=0}^{a} \frac{(k-1)q!(a+b-q)!}{(ka+b)!} \frac{1}{(a-q)!} \]

\[ = \sum_{q=0}^{a} \frac{kq!(a+b-q)!}{(ka+b-q)!} \frac{1}{(a-q)!} \]

\[ = \sum_{q=0}^{a} \frac{kq!(ka+b-q)!}{(ka+b)!} \frac{1}{(a-q)!} \]

we have

(13.16) \[ P\{E = 0 \mid D, T \in J_D \} = \frac{b}{ka+b} \]

If \(D = \emptyset\), where \(\emptyset\) is the empty set, \(J_D = J\), and \(b = 0, a = r\). For any \(T \in J\), \(E_D\) is the total number of cyclical elements in \(X\). Hence

(13.17) \[ P\{E_D = r \} = \frac{b^{r-1}((k-1)r-1)!}{(kr-1)!} \frac{1}{(r-1)!} \]

The distribution of the number of structures is given by

(13.18) \[ P\{m = 1\} = \sum_{j=1}^{r} P\{m = 1, E_D = j\} \]

\[ = \sum_{j=1}^{r} P\{m = 1, E_D = j\} P\{E_D = j\} \]

\[ = \sum_{j=1}^{r} \frac{k^{j-1}((k-1)(r-1)!)(kr-1-1)!}{(kr-1)!} \alpha(1, j) \]

\[ 1 \leq 1 \leq r \]
where \( \alpha(i,j) \) is the probability of \( i \) cycles in a permutation of \( j \) elements.

Since \( \alpha(1,j) = 1/j \)

\[
P\{m = 1\} = \sum_{j=1}^{n} \frac{k^{j-1}(k-1)(r-1)!}{(k-1)!(r-j)!}
\]

For any pair \((x,T)\), \( S_T(x) \) has been defined as the set of \( x \) containing elements, the number of elements in \( S_T(x) \), as the set of the structure containing \( x \). The possible values of \( c \) are \( k, 2k, \ldots, rk \), for suppose \( S_T(x) \) contains \( \nu \) elements from \( A_k \) \((1 \leq \nu \leq r)\). Then, since \( S_T(x) \) is the set of immediate predecessors of these \( \nu \) elements, \( c = \nu k \).

Let \( X_{jk} \) be a subset of \( X \) containing \( jk \) elements. We have

\[
P\{c = jk\} = \text{(number of ways in which } X_{jk} \text{ can be chosen)} \times P\{X_{jk} \text{ forms the picked structure}\}
\]

\[
= \binom{kr}{j} \cdot P\{z \in X_{jk}\} \cdot P\{T \subseteq X_{jk}\}
\]

\[
\quad \cdot P\{T(X-X_{jk}) \subseteq X-X_{jk}\} \cdot P\{m-1|n-jk\}
\]

\[
= \frac{(kr)!}{(kj)!} \cdot \frac{k!}{j!(k-1)!(r-j)!(r-j-k+1)!}
\]

\[
\quad \cdot \sum_{t=1}^{j} \frac{k^{t-1}(t-1)!}{(k-1)!(r-t)!}
\]

\[
= \frac{(kr)!}{(kj)!} \cdot \frac{k!}{j!(k-1)!(r-j)!(r-j-k+1)!}
\]

We obtain the joint distribution of six-length and cycle length directly from (13.8) and (13.9) by letting \( D = \emptyset \). Thus, we obtain

\[
P\{s = j, \ell = q, q < j\} = \frac{(kr)!k^{q-2}(r-1)!(r-j+1)!}{(kr)!!(r-j+2)!}
\]

\[\text{for } 2 \leq j \leq r+1\]

and

\[
P\{s = j, \ell = j\} = \frac{k^{j-1}(r-1)!}{(kr)!!(r-j)!}
\]

\[\text{for } 1 \leq j \leq r\].
From these we obtain the marginal distributions

\[(13.23) \quad P\{l = q\} = \frac{k^{q-1}(r-1)!(kr-q)!}{(kr)!(r-q)!} \sum_{j=q}^{r} \frac{(k-1)^{j-2}(r-1)!(kr-1)!}{(kr)!(r-j+1)!} \]

and

\[(13.24) \quad P\{s = j\} = \frac{1}{kr} \]

\[(13.25) \quad P\{s = j\} = \frac{(k-1)(j-1)k^{j-2}(r-1)!(kr-1)!}{(kr)!(r-j+1)!} \quad 2 \leq j \leq r+1 \]

In determining the distribution of the number of predecessors, we recall that the selected element \( x \) is counted as a predecessor of itself. The possible values of \( p \), therefore, are \( 1, k+1, \ldots, (r-1)(k+1) \), if \( x \) is not a cyclical element in \( T \), and \( k, 2k, \ldots, rk \), if \( x \) is cyclical. It will be noted that if \( p = 1 \), the selected element \( x \) belongs to \( A_0 \), while if \( p > 1 \), \( x \in A_k \) in the chosen transformation. Hence,

\[(13.26) \quad P\{p = 1\} = P\{s \in A_0\} = \frac{k^{r-1}}{k} \]

As before, let \( I_{jk} \) be a subset of \( X \) containing \( jk \) elements. Then

\[(13.27) \quad P\{p = jk\} = \text{number of ways in which } I_{jk} \text{ can be chosen} \]

\[= P\{TX_{jk} \subseteq I_{jk}\} \cdot P\{T(X-I_{jk}) \subseteq X-I_{jk}\} \]

\[= P\{n = 1, x \in X, (x)\mid n = jk\} \]

\[= \frac{(k^r)}{(k^r)!} \cdot \frac{(k^r)(k^r)\cdots (k^r)\cdots (k^r)}{(k^r)!} \cdot \frac{1}{j!(j(k-1))!(r-j)!(k^r)!} \]

\[= \frac{1}{r^k} \cdot \sum_{k=1}^{r} \frac{1}{j \cdot \frac{1}{r^k}} \cdot P\{p = j\} \]

\[= \frac{(r-1)!(r(k-1))!(k-1)!(k^r)!}{(j-1)!(j(k-1))!(r-j)!(r-j)(k-1)!}(rk)! \]
To obtain \( P\{p = jk+1\} \), we consider the following: Let \( X^p \) be a set of \( jk+1 \) elements, and let \( x^p \) be a given element of \( X^p \). Consider the set of all transformations of \( X^p \), say \( J^p \), such that \( x^p \) is mapped into itself and in addition has \( k \) immediate predecessors in \( X^p-x^p \) while each element in \( X^p-x^p \) has either zero or \( k \) immediate predecessors. The total number of such transformations is

\[
\frac{(jk)!(jk)l}{(j-1)!(j(k-1)l!k)!}
\]

Let \( n_x^p \) be the total number of cyclical elements in \( X^p-x^p \). Then, in a manner similar to that used for the sets \( D \) and \( J^p_D \), we find that

\[
P\{n_x^p = q|x^p, T \in J^p\} = \frac{(k-1)!q(k-1)!q(jk-q)j}{(jk)!(j-1)q!}
0 \leq q \leq j-1
\]

so that

\[
P\{n_x^p = 0|x^p, T \in J^p\} = \frac{1}{j}
\]

Let us now return to the sets \( X \) and \( J \), and for any pair \( (x,T) \), let \( P_T(x) \) denote the set of predecessors of \( x \). Then

\[
(13.27) \quad P\{p = jk+1\} = \frac{\{y\}}{\{x\}} P\{x \in J\} \times P\{x \in X\} \times P\{x \in X\} \times P\{x \in X\}
\]

\[
= \frac{(jk)!(jk)l}{(j-1)!(j(k-1)l!k)!} P\{x \in X\} \times P\{x \in X\} \times P\{x \in X\}
\]

\[
\cdot P\{x \in J\} \times P\{x \in X\} \times P\{x \in X\}
\]

\[
= \frac{(jk-1)!}{(j-1)!(j(k-1)s-1)!} \cdot \frac{(k+r-1)!}{(r-j)!((k-1)(r-j)-1)!} \cdot \frac{(k+r-1)!}{(k-1)!r(k-1)!s-1}\]

\[
= \frac{(jk-1)!}{(j-1)!(j(k-1)s-1)!} \cdot \frac{(k+r-1)!}{(r-j)!((k-1)(r-j)-1)!} \cdot \frac{(k+r-1)!}{(k-1)!r(k-1)!s-1}\]

\[
= \frac{(jk-1)!}{(j-1)!(j(k-1)s-1)!} \cdot \frac{(k+r-1)!}{(r-j)!((k-1)(r-j)-1)!} \cdot \frac{(k+r-1)!}{(k-1)!r(k-1)!s-1}\]

for \( j = 0,1,\ldots,r-1 \).
Asymptotic approximations for the various distributions can be found as follows. For fixed \( k \), as \( r \) becomes large, we have

\[
P\{X^{-1} = j\} \sim \frac{(k-1)^{\frac{1}{2}}(k-r)^{\frac{1}{2}}}{r^{\frac{1}{2}}(r-j)^{\frac{1}{2}}} \frac{kr-j-\frac{1}{2}}{kr-j-\frac{1}{2}}
\]

\[
j(x) = \frac{k}{2} - \frac{1}{2} \left( \frac{k-1}{kr} \right) j^2
\]

The asymptotic density of \( \frac{(k-1)^{1/2}}{kr} \) is

\[
P(x) = xe^{-\frac{1}{2}x^2}
\]

Given that \( \theta = j \), the conditional distribution of \( m \), the number of structures, is asymptotically normal as \( j \) becomes large with

\[
E(m|j) \sim \log j + \gamma \cdot \frac{1}{2j}
\]

where \( \gamma \) is Euler's constant and

\[
\text{Var}(m|j) \sim \log j + \gamma \cdot \frac{\pi^2}{6} + \frac{3}{2j} - \frac{1}{2j^2}
\]

The marginal distribution of \( m \) is asymptotically normal with expected value and variance each approximately equal to \( \frac{1}{2} \log \left( \frac{kr}{k-1} \right) \).

The asymptotic distribution of structure size is given by
\[ (13.32) \quad P\{ s = j, \ell = q, q < j \} \sim \frac{(k-1)^{j-2} e^{-\frac{1}{2}kr} e^{-\frac{1}{2}r(j-1)}}{(kr)^{j-1}} \]

\[ \frac{1}{r^{\frac{1}{2}}} e^{-\frac{1}{2}r} e^{-\frac{1}{2}r(j-1)} (kr)^{j-\frac{3}{2}} \]

\[ \frac{1}{2kr^{\frac{1}{2}}} (r-j)^{1/2} \]

the asymptotic density of \( x = \frac{q}{kr} \) is

\[ P(x) = \frac{1}{2(1-x)^{1/2}} \quad 0 \leq x \leq 1 \]

From (13.21) we obtain, as \( r \) becomes large

\[ (13.33) \quad P\{ s = j, \ell = q, q < j \} \sim \frac{(k-1)^{j-2} e^{-\frac{1}{2}kr} e^{-\frac{1}{2}r(j-1)}}{(kr)^{j-1}} \frac{1}{r^{\frac{1}{2}}} e^{-\frac{1}{2}r} e^{-\frac{1}{2}r(j-1)} (kr)^{j-\frac{3}{2}} \]

\[ \frac{1}{2kr^{\frac{1}{2}}} (r-j)^{1/2} \]

\[ \frac{1}{2} \frac{(k-1)}{kr} (j-1)^2 \]
From (13.22), we obtain

\[ P \{ s = j, f = j \} \sim \frac{1}{\text{erf}^2} \left( \frac{1}{2} \right) \frac{1}{\left( k - 1 \right)^{2}} \frac{1}{\left( r - 1 \right)^{2}} \frac{1}{\left( k - 1 \right)^{2}} \frac{1}{\left( r - 1 \right)^{2}} \]

\[ \sim \frac{1}{kr} \left( k - 1 \right)^{2} \frac{1}{kr} \]

From these, we have

\[ P \{ s = j \} \sim \frac{1}{k} \left( k - 1 \right)^{2} \frac{j}{2} \]

or the asymptotic density of \( t = \frac{(s-1)}{kat} \) is \( p(t) = (k-1)te^{-\frac{1}{2}(k-1)t^2} \). Also

\[ P \{ \ell = q \} \sim \int_{q}^{\infty} \left( \frac{k-1}{kr} \right) e^{-\frac{1}{2}(k-1)j^2} dj \]

\[ = \frac{2\pi(k-1)^{1/2}}{kr} e^{-\left( \frac{k-1}{kr} \right)^{1/2} \Phi^{-1} \left( \frac{k-1}{kr} \right)^{1/2} q} \]

where \( \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \).

The asymptotic distribution of predecessors can be found directly from (13.26) and (13.27) with the use of Stirling's approximations. In this manner we obtain

\[ P \{ p = jk \} \sim \frac{1}{\left( j! \right)^{1/2} \left( k! \right)^{1/2} \left( r-1 \right)^{1/2}} \rightarrow 0 \quad \text{as} \; r \rightarrow \infty \]

and

\[ P \{ p = jk+1 \} \sim \frac{1}{\left( j! \right)^{1/2} \left( k! \right)^{1/2} \left( r-1 \right)^{1/2}} \rightarrow 0 \quad \text{as} \; r \rightarrow \infty \]

These results can also be obtained in the following manner. Since \( \sum_{j=1}^{r} P \{ p = jk \} \)

- probability that an element selected at random is cyclic, and this probability approaches zero as \( r \) becomes large, it is clear that \( P \{ p = jk \} \rightarrow 0 \) as \( r \rightarrow \infty \).

Writing for simplicity \( P \{ p = jk+1 \} = \phi_{jk+1} \), we have
(13.39) \[ \psi_k = \frac{k-1}{k} \psi_{k-1} \]
\[ \psi_{k-1} = \frac{k}{k} \psi_k \]
\[ \vdots \]
\[ \psi_{j,k-1} = \frac{k}{k} \sum_{i=1}^{k} \psi_{i,k-1} \psi_{2,k-1} \psi_{3,k-1} \cdots \psi_{k,k-1} \]
\[ \sum_{t=1}^{k} r_t = j-1 \]
\[ \vdots \]

Consider the function \( y = h(x) = \sum_{t=0}^{\infty} x^{t+1} \psi_{t+1} \). Then

(13.40) \[ y = h(x) = (x^{L+1} + \frac{1}{k}(h(x))^k) = xg(y) \]

It follows that

(13.41) \[ \psi_{j,k-1} = \frac{1}{(j,k)!} \left( \frac{d}{dy} \left( g(y) \right)^{j,k+1} \right) \bigg|_{y=0} \]
\[ = \frac{(jk)!}{j!(j(k-1)+1)!} \frac{(k-1)^{j(k-1)+1}}{k^{j,k+1}} \]

the result obtained in (13.36). When \( j \) is large but still small relative to \( r \), we have

(13.42) \[ F \left( p-j,k-1 \right) \sim \frac{\left( e^{-j,k+1} \right)^{j,k+1} \frac{1}{2(j,k)+1} \left( k-1 \right)^{j(k-1)+1}}{\sqrt{2\pi e} \cdot j^{j,k+1} \int \frac{\left( e^{-j,k+1} \right)^{j,k+1} \frac{1}{2(j,k)+1} \left( k-1 \right)^{j(k-1)+1}}{\sqrt{2\pi e} \cdot j^{j,k+1} \int \frac{1}{\sqrt{2\pi j^{3/2}(k-1)^{1/2}k^{1/2}}} \right)} \]

14. Case of each element having at most \( k \) predecessors. Let \( I \) be the set of transformations of \( X \) into \( Y \) such that each element \( x \in X \) has at most \( k \) immediate predecessors where \( k \) is a given number. For a given \( T \in I \), let \( A_i \) be the set of elements with exactly \( i \) immediate predecessors \( (i=0,1,2,\ldots,k) \), and let \( B_j \) be the set of elements of \( X \) such that \( \Theta_j = A_j \) \((j=1,2,\ldots,k) \). Clearly
Also, if \( a_i \) is the number of elements in \( A_i \), then \( i a_i \) is the number of elements in \( B_i \), and
\[
\sum_{i=0}^{k} a_i - \sum_{i=1}^{k} i a_i = n.
\]

Let \( \mathbf{N} = (n_0, n_1, \ldots, n_k) \) be a set of \((k+1)\) integers satisfying \((14.2)\). Then
\[
J = \bigcup_{\mathbf{N}} J_{\mathbf{N}}
\]
where \( J_{\mathbf{N}} \) is the set of all transformations \( \mathbf{T} \in J \) for which \( a_0 = n_0, a_1 = n_1, \ldots, a_k = n_k \).

The number of transformations in \( J_{\mathbf{N}} \) is
\[
Q(\mathbf{N}) = \text{number of ways in which the sets } A_0, A_1, \ldots, A_k \text{ can be chosen} \times \text{number of ways in which the sets } B_1, B_2, \ldots, B_k \text{ can be chosen} \times \text{(number of transformations mapping } B_i \text{ onto } A_i \text{ such that each element of } A_i \text{ has exactly } i \text{ predecessors)}
\]

Or
\[
Q(\mathbf{N}) = \frac{n!}{n_0! n_1! \ldots n_k!} \cdot \frac{n!}{n_1(2n_2)! \ldots (kn_k)!} \cdot \frac{(2n_2)!}{(2i)_2} \ldots \frac{(kn_k)!}{(k!)^n_k}
\]

The total number of transformations in \( J \) is given by \( \sum_{\mathbf{N}} Q(\mathbf{N}) \). To obtain the asymptotic value of this sum as \( n \) becomes large, we proceed as follows: If we approximate \( n_1 \) by \( \sqrt{2\pi n_1 e^{-n_1}} \) and take the logarithm of \( Q(\mathbf{N}) \), we obtain
\[
\log Q(\mathbf{N}) = 2\log n! - \frac{(k+1)}{2} \log 2\pi - \sum_{i=0}^{k} \left( n_i \cdot \frac{1}{2} \right) \log n_i
\]
Since $N^*$ satisfies (14.2), i.e.,

$$\sum_{i=0}^{k} n_i = \sum_{i=1}^{k} in_i = n$$

we can write

$$n_o = \sum_{i=2}^{k} (i-1)n_i \quad , \quad n_1 = n - \sum_{i=2}^{k} in_i$$

and

(14.7) \hspace{1cm} \log \varphi(N^*) = \Psi(n_2, n_3, \ldots, n_k), \quad \text{say.}

Neglecting the terms $-\frac{1}{2} \log n_i$ ($i=0, 1, \ldots, k$) and differentiating $\Psi$ with respect to $n_j$ ($j=2, \ldots, k$), we obtain

(14.8) \hspace{1cm} \frac{\partial \Psi}{\partial n_j} = -(j-1)\log n_o - (j-1) \cdot j \log n_1 \cdot j - \log n_j - \log j! = -1

\hspace{1cm} \log \left( \frac{n_1}{n_o n_j j!} \right)

Also

(14.9) \hspace{1cm} \frac{\partial^2 \Psi}{\partial n_1 \partial n_j} = - \frac{(i-1)(j-1)}{n_o n_1} - \frac{i1}{n_j} - \frac{\delta_{ij}}{n_1}

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$.

Let $n_o^*, n_1^*, \ldots, n_k^*$ be values of $n_0, n_1, \ldots, n_k$, respectively, which maximize $\Psi$, i.e., $n_o^*, n_1^*, \ldots, n_k^*$ are a set of values satisfying the $(k+1)$ equations

$$\frac{\partial \Psi}{\partial n_j} = 0 \quad j=2, \ldots, k$$

$$\sum_{i=0}^{k} n_i = n$$

$$\sum_{i=1}^{k} in_i = n.$$ 

From (14.8) we obtain

(14.10) \hspace{1cm} n_j^* = \frac{n_o^*}{j!} \left( \frac{n_o^*}{n_o^*} \right)^j \quad (j=2, \ldots, k).

It will be observed that (14.10) holds trivially for $j=0$ and $1$.

Let $n_o^* = \alpha n$ and $n_1^* = \alpha \beta n$. Then

(14.11) \hspace{1cm} n_j^* = \frac{\alpha \beta^j}{j!} \quad (j=0, 1, \ldots, k).
It is easy to see that the desired value of $\beta$ is the single positive real root of (14.12), say, $\beta^*$. The value of $\beta^*$ decreases as $k$ increases, with $\beta^* = \sqrt{2}$ when $k = 2$, and $\beta^* = 1$ when $k = \infty$. From (14.11) we obtain that

\begin{equation}
(14.14) \quad n_j^* = \frac{n(\beta^*-1)k!}{\beta^{k+1}}.
\end{equation}

Let

\begin{equation}
(14.15) \quad q = (q_{ij}) = \left( -\frac{\partial^2 \psi}{\partial \alpha_{n_1+1} \partial \alpha_{n_j+1}} \right)_{n_0-n_0^*, n_1-n_1^*, \ldots, n_k-n_k^*}
\end{equation}

Then

\begin{equation}
(14.16) \quad Q(N^*) \sim \min_{\mathbb{R}^k} \frac{\prod_{i=0}^{k-1} \frac{k-1}{2} \sum_{j=1}^{k-1} q_{ij} (n_{j+1} - n_{j+1}^*) (n_{j+1} - n_{j+1}^*)}{(2\pi)^{k/2} \prod_{i=0}^{k-1} n_i \Gamma(1/2)} \prod_{i=1}^{k-1} n_{i+1} \Gamma(1/2)
\end{equation}

and

\begin{equation}
(14.17) \quad \sum_{N^*} Q(N^*) \sim \min_{\mathbb{R}^k} \frac{n + \frac{1}{2} \prod_{i=0}^{k-1} \frac{k}{2} \sum_{j=1}^{k-1} q_{ij} (n_{j+1} - n_{j+1}^*) (n_{j+1} - n_{j+1}^*)}{(2\pi)^{k/2} \prod_{i=0}^{k-1} n_i \Gamma(1/2)} \prod_{i=1}^{k-1} n_{i+1} \Gamma(1/2)
\end{equation}

\begin{equation}
- \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} q_{ij} (n_{j+1} - n_{j+1}^*) (n_{j+1} - n_{j+1}^*) \prod_{i=2}^{i=1} dn_2 \ldots dn_k
\end{equation}

\begin{equation}
- \frac{n \prod_{i=0}^{k-1} \Delta_i^{1/2} \prod_{i=1}^{k} n_i \Gamma(1/2)}{(2\pi)^{1/2} \prod_{i=0}^{k-1} n_i \Gamma(1/2)}
\end{equation}
Let \( \xi' \) and \( \rho' \) denote the row vectors \((1,2,\ldots,k-1)\) and \((2,3,\ldots,k)\), respectively, and let \( D \) be the \((k-1) \times (k-1)\) diagonal matrix

\[
D = \begin{pmatrix}
\frac{1}{n_2} & 0 & \cdots & 0 \\
0 & \frac{1}{n_3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{n_k}
\end{pmatrix}
\]

Then

\[
|Q| = \begin{vmatrix}
1 & 0 & \frac{1}{n_2} \xi' \\
0 & 1 & \frac{1}{n_3} \rho' \\
-\frac{1}{n_2} & -\frac{1}{n_3} & D
\end{vmatrix}
= \left| D \right| \left| \frac{1}{n_2} \xi' D^{-1} \xi' - \frac{1}{n_3} \rho' D^{-1} \rho' \right|
= \left| D \right| \left| \frac{1}{n_2} \xi' D^{-1} \xi' - \frac{1}{n_3} \rho' D^{-1} \rho' \right|
\]

\[
= \frac{1}{n_2} \sum_{j=2}^{k} (j-1)^2 n_j^2 \frac{1}{n_3} \sum_{j=2}^{k} j(j-1) n_j^2
\]

\[
= \frac{1}{n_2} \sum_{j=2}^{k} j(j-1) n_j^2 \frac{1}{n_3} \sum_{j=2}^{k} j^2 n_j^2
\]

\[
= n \left( \sum_{j=2}^{k} j(j-1) n_j^2 \right) \prod_{i=0}^{k-1} \frac{1}{n_i^2}
\]

Hence

\[
\sum_{n=0}^{\infty} q(n) \sim \frac{n \ln n}{(2\pi)^{k} \prod_{j=2}^{k} j(j-1) n_j^2 \frac{1}{n_3} \sum_{j=2}^{k} j^2 n_j^2}
\]

From (14.11), (14.12), and (14.13), we have

\[
\sum_{j=2}^{k} j(j-1) n_j^2 - \sum_{j=2}^{k} j(j-1) \frac{\alpha \sigma^n_{k j}}{j!} - \rho \alpha n \sum_{j=1}^{k-1} \frac{\beta_{j}^n}{j!} - \beta n(1 - \frac{\alpha \sigma^n_{k k}}{(k-1)!})
\]

\[
= n(k-1) \beta^n
\]
Since (14.19) is necessarily positive, this yields \( 1 - \frac{1}{k-1} \) as an upper bound for \( \beta^* \). Also

\[
1 = \prod_{i=0}^{k} \left( \frac{n_i!}{n!} \right)^{\frac{1}{k}} = \prod_{i=0}^{k} \left( \alpha \beta_{1-n}^{*} \right)^{\frac{1}{k}} = (\alpha \beta_{1-n}^{*})^n = \frac{n(\beta_{-1}^*)^n}{\beta_{k-1}^*} \\

so that

\[
\sum_{n=1}^{\infty} q(n^*) = \frac{n\beta_{k-1}^*}{(2\pi n)^{1/2}(k-(k-1)\beta^*)^{1/2}(\beta^* - 1)^n} = \frac{e^{-n\beta_{k-1}^*}}{(k-(k-1)\beta^*)^{1/2}(\beta^* - 1)^n}
\]
correct to terms within order 1/n.

It is easy to see that the variables \( \frac{n_0}{n}, \frac{n_1}{n}, \ldots, \frac{n_k}{n} \) have a limiting singular multivariate normal distribution with expected values \( \frac{n_0}{n}, \frac{n_1}{n}, \ldots, \frac{n_k}{n} \).

The non-singular covariance matrix of the limiting distribution of

\[
\frac{n_0}{n}, \frac{n_1}{n}, \ldots, \frac{n_k}{n}
\]
is

\[
\sum_{i=2}^{k} \left( \delta_{ij} \frac{n_i}{n} - \frac{n_i}{n} \frac{n_j}{n} \right) = \left( \delta_{ij} \frac{\alpha \beta_{k-1}^*}{i!} - \frac{(i-1)(i-1)}{i!} \frac{\alpha \beta_{k-1}^*}{i!} \right) = \left( \frac{(i-1)(i-1)}{i!} \right)

It is clear that the variables \( \frac{n_0}{n}, \frac{n_1}{n}, \ldots, \frac{n_k}{n} \) converge stochastically to their expected values.

When \( k = 2 \), we have \( n_0 = n_2, n_1 = n-2n_2, \beta^* = \sqrt{2} \), and

\[
\frac{n_0}{n} = \frac{1}{2} (2-\sqrt{2})n \\\n\frac{n_1}{n} = (\sqrt{2}-1)n \\\n\frac{n_2}{n} = \frac{1}{2} (2-\sqrt{2})n
\]

with
\( \Phi(n) = \frac{n! (2+\frac{1}{2})^{n + \frac{1}{2}}}{\sqrt{\pi n^2} n^2} \)

correct to within terms of order \( \frac{1}{n} \). The asymptotic distribution of \( \frac{\text{order}}{n} \) is normal with expected values \( \frac{1}{2} (2+\sqrt{2}) \sqrt{n} \) and variance \( \frac{1}{4} (2+\sqrt{2})^2 \).

As before, for any pair \((x,T)\) let \( s \) denote the six-length of \( x \) and \( l \) the cycle length. Also, let \( S \) be the set of elements in the six of \( Tx \), i.e., \( S = (Tx,Tx^2,...) \), and let \( a_i \) be the number of elements in \( S \cap \Lambda_i \) \((i=1,2,...,k)\), Clearly, \( \sum_{i=1}^{k} a_i = s-1 \). Then

\[
(14.25) \quad P\left\{ s-x, l=q, q< r, a_1 = \gamma_1, ..., a_k = \gamma_k, x \in A_0, T^{r-1-q} x \in A_1 \right\}
\]

\[
= \left( \text{number of ways in which the sets } A_0, A_1, ..., A_k \text{ can be chosen} \right) \times \left( \text{number of ways in which } S \cap A_1, S \cap A_2, ..., S \cap A_k \text{ can be chosen} \right) \times \left( \text{number of ways in which the remaining elements of the sets } B_1, B_2, ..., B_k \text{ can be chosen} \right) \times \sum_{t=1}^{k} \left( \text{number of ways in which the remaining predecessors for the elements of } A_t \text{ can be chosen from these remaining elements of } B_t \right)
\]

\[
= \frac{n!}{n_0 n_1! ... n_k!} \cdot \frac{n_1! ... n_k!}{(n_1-\gamma_1)! ... (n_k-\gamma_k)!} \cdot \gamma_1^{(r-2)} \cdot \gamma_1^{(r-1)} \times \gamma_2^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-3)} ... \gamma_k^{(r-3)} ...
\]

\[
= \frac{n!}{n_0 n_1! ... n_k!} \cdot \frac{n_1! ... n_k!}{(n_1-\gamma_1)! ... (n_k-\gamma_k)!} \cdot \gamma_1^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-3)} ... \gamma_k^{(r-3)} ...
\]

\[
= \frac{n!}{n_0 n_1! ... n_k!} \cdot \frac{n_1! ... n_k!}{(n_1-\gamma_1)! ... (n_k-\gamma_k)!} \cdot \gamma_1^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-3)} ... \gamma_k^{(r-3)} ...
\]

\[
= \frac{n!}{n_0 n_1! ... n_k!} \cdot \frac{n_1! ... n_k!}{(n_1-\gamma_1)! ... (n_k-\gamma_k)!} \cdot \gamma_1^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-3)} ... \gamma_k^{(r-3)} ...
\]

\[
= \frac{n!}{n_0 n_1! ... n_k!} \cdot \frac{n_1! ... n_k!}{(n_1-\gamma_1)! ... (n_k-\gamma_k)!} \cdot \gamma_1^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-1)} \gamma_2^{(r-3)} ... \gamma_k^{(r-2)} \gamma_1^{(r-3)} ... \gamma_k^{(r-3)} ...
\]
It follows that

\[(14.26) \quad P\{s=r, l=q, q<r, t=x_1, \ldots, x_k = \gamma_k, X \in \mathbb{N}_0|\mathbb{S}\} = \frac{n}{n!} \frac{(n-r)!}{(n-r)!} \sum_{i=2}^{k} \frac{n_i!}{(n_i)!} \frac{\gamma_t^k}{\gamma_t^t (n_i - \gamma_t)!} \]

Similarly, we find

\[(14.27) \quad P\{s=r, l=q, q<r, t=x_1, \ldots, x_k = \gamma_k, X \in \mathbb{N}_0|\mathbb{S}\} = \frac{j((n_j - \gamma_j)}{n} \frac{(n-r)!}{(n-r)!} \sum_{i=2}^{k} \frac{n_i!}{(n_i)!} \frac{\gamma_t^k}{\gamma_t^t (n_i - \gamma_t)!} \]

so that

\[(14.28) \quad P\{s=r, l=q, q<r, t=x_1, \ldots, x_k = \gamma_k, X \in \mathbb{N}_0|\mathbb{S}\} = \frac{(n-r)!}{n!} \frac{(n-r)!}{(n-r)!} \sum_{i=2}^{k} \frac{(n_i)!}{(n_i)!} \frac{\gamma_t^k}{\gamma_t^t (n_i - \gamma_t)!} \]

and

\[(14.29) \quad P\{s=r, l=q, q<r, t=x_1, \ldots, x_k = \gamma_k, X \in \mathbb{N}_0|\mathbb{S}\} = \frac{(n-r)!}{n!} \frac{(n-r)!}{(n-r)!} \sum_{i=1}^{k} \frac{\gamma_t^k}{\gamma_t^t (n_i - \gamma_t)!} \frac{n_i!}{(n_i)!} \frac{\gamma_t^k}{\gamma_t^t (n_i - \gamma_t)!} \]

To find the joint marginal probability distribution of \(s\) and \(l\), it is necessary to sum (14.29) and (14.30) over all values of the \(\gamma_i\) such that \(0 \leq \gamma_i \leq n_i, i=1,2,\ldots,k, \sum_{i=1}^{k} \gamma_i = r-1\), and to take the expected value of each sum with respect to \(\mathbb{S}\). To obtain the asymptotic value of \(P\{s=r, l=q, q<r\}\) we replace each factorial in (14.29) by the first term of its Stirling's approximation and consider the logarithms of the resulting expression as a function of \(\gamma_2, \ldots, \gamma_k\), namely
Neglecting the terms \( \log \left( \sum_{i=2}^{k} (\gamma_i - 1) \right) \), \(-\frac{1}{2} \log \gamma_t\), and \(-\frac{1}{2} \log (n_t - \gamma_t)\) which are relatively very small as \( n \) becomes large, and differentiating \( g \) with respect to \( \gamma_i\), \( i = 2, \ldots, k\), we obtain
\[
(14.32) \quad \frac{\partial g}{\partial \gamma_i} = \log i - \log \gamma_i - \log(n_i - \gamma_i) - \log(n_1 - \gamma_1) - \log \left( \frac{n_1}{\gamma_1 (n_1 - \gamma_1)} \right).
\]

Also
\[
(14.33) \quad \frac{\partial^2 g}{\partial \gamma_i \partial \gamma_j} = -\frac{\delta_1}{\gamma_i} - \frac{\delta_1}{i - \gamma_i} - \frac{1}{\gamma_i} - \frac{1}{n_i - \gamma_i} - \left( \frac{\delta_1 n_1}{\gamma_1 (n_1 - \gamma_1)} + \frac{n_1}{\gamma_1 (n_1 - \gamma_1)} \right).
\]

Let \( \gamma_1^*, \gamma_2^*, \ldots, \gamma_k^* \) be the values of \( \gamma_1, \gamma_2, \ldots, \gamma_k \), respectively, satisfying the \( k \) equations
\[
\sum_{i=1}^{k} \gamma_i = n-1 \quad \text{and} \quad \frac{\partial g}{\partial \gamma_i} = 0 \quad i = 2, \ldots, k.
\]

From (14.32) we have
\[
(14.34) \quad \gamma_i^* = \frac{\ln n_i \gamma_i^*}{n_i + (i-1) \gamma_i^*}.
\]

It will be observed that (14.34) is satisfied trivially when \( i = 1 \).

Let
\[
(14.35) \quad q^* = (q_{ij}^*) = \left( \begin{array}{c} \text{(continued)} \end{array} \right)
\]
\[
- \left( \begin{array}{c} \text{(continued)} \end{array} \right)
\]
\[
- \left( \begin{array}{c} \text{(continued)} \end{array} \right)
\]
Let \( \frac{n_i}{Y_i(n_i - Y_i)} = \theta_i \). Then

\[
| \mathbf{q}^* | = \begin{bmatrix}
\theta_1 & \theta_2 & \cdots & \theta_k
\end{bmatrix}
\begin{bmatrix}
\phi_1 \phi_1 \cdots \phi_k
\end{bmatrix}
= \begin{vmatrix}
1 & \phi_1^{1/2} & \phi_1^{1/2} & \cdots & \phi_1^{1/2}
-\phi_1^{1/2} & \theta_2 & 0 & \cdots & 0
-\phi_1^{1/2} & 0 & \theta_3 & \cdots & 0
\vdots & \vdots & \vdots & \ddots & \vdots
-\phi_1^{1/2} & 0 & 0 & \cdots & \theta_k
\end{vmatrix}
\]

\[
- \prod_{t=2}^{k} \theta_t \left( 1 + \sum_{i=2}^{k} \frac{\theta_i}{\theta_1} \right) = \sum_{i=1}^{k} \frac{Y_i(n_i - Y_i)}{n_i} \prod_{t=1}^{k} \frac{n_t}{Y_t(n_t - Y_t)}
\]

But

\[
P \{ s = r, \ell = q, \varrho < x \} \sim \frac{(n-r+1)^{-\frac{1}{2}} (r-1)^{-\frac{3}{2}} \left( \sum_{i=2}^{k} (i-1) Y_i^* \right) \prod_{t=1}^{k} n_t}{n^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \frac{k}{2} \left( \frac{Y_t^* \phi_t^{1/2}}{2(n_t - Y_t)} \right)}
\]

Substituting the value of \( Y_i^* \) \( i = 2, \ldots, k \) from (14.34) we obtain

\[
P \{ s = r, \ell = q, \varrho < x \} \sim \frac{\prod_{i=2}^{k} \phi_i^{1/2} \phi_i^{1/2} \prod_{t=1}^{k} n_t}{n^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \frac{k}{2} \left( \frac{Y_t^* \phi_t^{1/2}}{2(n_t - Y_t)} \right)}
\]
\[
\begin{align*}
\text{The function } \gamma_1^* & = \frac{(r-1)(\frac{n}{n-1})}{r-2} \left( \sum_{t=1}^{k} t(1-\frac{\hat{\beta}^*}{\hat{\beta}^*}) \frac{\hat{\beta}^*}{n} \right) \left( 1 - \frac{\hat{\beta}^*}{n} \right) \\
& = \left( \frac{(r-1)(\frac{n}{n-1})}{r-2} \left( \sum_{t=1}^{k} t(1-\frac{\hat{\beta}^*}{\hat{\beta}^*}) \frac{\hat{\beta}^*}{n} \right) \right) \left( 1 - \frac{\hat{\beta}^*}{n} \right)
\end{align*}
\]

Or

\[
(14.38) \quad P\{s=r, \, l=q, q<r \mid n^* \} \sim \frac{(r-1)(\frac{n}{n-1})}{r-2} \left( \sum_{t=1}^{k} t(1-\frac{\hat{\beta}^*}{\hat{\beta}^*}) \frac{\hat{\beta}^*}{n} \right) \left( 1 - \frac{\hat{\beta}^*}{n} \right)
\]

But

\[
(r-1) = \gamma_1^* \sum_{t=1}^{k} \frac{t}{n^*} \sim \gamma_1^* \sum_{t=1}^{k} \frac{t}{n} \sim \gamma_1^* \sum_{t=1}^{k} \frac{t}{n}
\]

Hence

\[
(14.39) \quad P\{s=r, \, l=q, q<r \mid n^* \} \sim \frac{1}{n} \left( \sum_{t=2}^{k} t(t-1) \frac{n^*}{n} \right) - \frac{(r-1)^2}{2n} \sum_{t=2}^{k} t(t-1) \frac{n^*}{n}
\]

However, as \( n \) becomes large, \( \sum_{t=2}^{k} t(t-1) \frac{n^*}{n} \) converges in probability to \( (k-2)\hat{\beta}^* \) so that

\[
(14.40) \quad P\{s=r, \, l=q, q<r \} = (k-2)\hat{\beta}^* \exp \left( -\frac{(k-1)^2}{2n} (k-2)\hat{\beta}^* \right)
\]

correct to terms of within order \( 1/n \).

Similarly, we find that

\[
(14.41) \quad P\{s=r, \, l=\ne\} = \frac{1}{n} \cdot \frac{(k-1)^2}{2n} (k-2)\hat{\beta}^*
\]

It follows that
(14.42) \[ P\{g = r\} = \left(\frac{k-1}{n}\right) (k-1) \beta^* \cdot \frac{1}{n} e^{-\frac{(k-1)^2}{2n} (k-1) \beta^*} \]

\[ \rightarrow \left(\frac{k-1}{n}\right) (k-1) \beta^* \cdot \frac{1}{n} e^{-\frac{(k-1)^2}{2n} (k-1) \beta^*} \]

as \( n \to \infty \). The asymptotic density of \( x = \frac{k-1}{n} \) is

(14.43) \[ p(x) = (k-1) \beta^* \cdot x \cdot e^{-\frac{x^2}{2} (k-1) \beta^*} \]

0 \leq x \leq \infty .

When \( k = 2 \), we have

(14.44) \[ p(x) = (2-\sqrt{2}) x \cdot e^{-\frac{x^2}{2} (2-\sqrt{2})} \]

It will be observed that the probability that an element selected at random

is cyclic is given by

(14.45) \[ \sum_{r=1}^{n} P\{g = r, q = r\} = \frac{1}{n} \int_{0}^{\infty} e^{-\frac{x^2}{2} (k-1) \beta^*} \]

\[ \sim \frac{1}{n} \int_{0}^{\infty} e^{-\frac{x^2}{2} (k-1) \beta^*} \]

\[ = \left(\frac{\pi}{2n(k-1) \beta^*}\right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty . \]

The asymptotic distribution of cycle length is given by

(14.46) \[ P\{l = q\} = \frac{1}{n} \cdot \frac{(k-1)^2}{2n} (k-1) \beta^* \cdot \sum_{r=q+1}^{n} \frac{1}{r} \cdot \frac{k(k-1) \beta^*}{n} \cdot \frac{(k-1)^2}{2n} (k-1) \beta^* \]

\[ \sim \left(\frac{k(k-1) \beta^*}{n}\right) \int_{q}^{\infty} e^{-\frac{x^2}{2} (k-1) \beta^*} \]

\[ \sim \left(\frac{2\pi(k-1) \beta^*}{n}\right)^{1/2} \frac{1}{\sqrt{q} (k-1) \beta^*} \frac{1}{2} \]
The asymptotic density of $h \cdot \frac{1}{\sqrt{n}}$ is

\[
(14.47) \quad p(h) = \frac{1}{(2\pi)^{1/2}} \left( \frac{n}{2} \right)^{1/2} \exp \left( -\frac{1}{2} \frac{h^2}{n} \right)
\]

Since

\[
(14.48) \quad \sum_{r=1}^{n} \frac{1}{r} P\{ \ell = j \} = \sum_{r=1}^{n} \frac{1}{r} P\{ Q_r = r \}
\]

where $Q_r$ is the total number of cyclical elements, we have

\[
(14.49) \quad \frac{1}{r} P\{ Q_r = r \} \sim \frac{1}{\left( \frac{n}{2} \right)^{1/2}} \exp \left( -\frac{1}{2} \frac{r^2}{n} \right) \left( \frac{r}{n} \left( k- \frac{2}{3} \right) \right)
\]

Or

\[
(14.50) \quad P\{ Q_r = r \} = \frac{1}{r} \left( \frac{r}{n} \right)^{1/2} \exp \left( -\frac{1}{2} \frac{r^2}{n} \right) \left( k- \frac{2}{3} \right)
\]

The expected value of $r$ is

\[
\left( \frac{\frac{n}{2} \left( k- \frac{2}{3} \right)}{\left( \frac{n}{2} \right)^{1/2}} \right)^{1/2} \text{ with variance } \left( \frac{n}{2} \right)^{1/2} \left( k- \frac{2}{3} \right) \left( \frac{n}{2} \right)^{1/2}
\]

Given that $Q_r = r$, the conditional distribution of $m$, the number of structures, is asymptotically normal as $r$ becomes large with

\[
(14.51) \quad E(m|r) \sim \log r + \gamma + \frac{1}{2r}
\]

where $\gamma$ is Euler's constant, and

\[
(14.52) \quad \text{Var}(m|r) \sim \log r + \gamma - \frac{r^2}{6} + \frac{1}{2r} - \frac{1}{2r^2}
\]

It follows that the marginal distribution of $m$ is asymptotically normal with expected value and variance given by

\[
(14.53) \quad E(m) \sim \frac{1}{2} \log n \quad \text{and} \quad \text{Var}(m) \sim \frac{1}{2} \log n
\]

The asymptotic distribution of structure size is given by
(14.54) \[ \mathbf{P}\{a = j\} = \text{(number of ways in which a set of } j \text{ elements can be chosen) \times \text{(Probability that this set forms the picked structure)} \]

\[
\gamma^n_j = \frac{1}{n} \frac{(n-j)!}{(n-j)^{n-j}} \frac{(n-j)!}{(n-j)^{n-j}} \frac{1}{n^{n-j}} \frac{1}{n^{j-1/2}} \frac{1}{n^{(n-j)^{n-j}}}
\]

The asymptotic density of \( x = a/n \) is given by

(14.55) \[ p(x) = \frac{1}{2(1-x)^{1/2}} \quad 0 \leq x \leq 1 \]

These asymptotic results are the same as the results for the general case of all transformations of \( X \) into \( X \).

To obtain the asymptotic distribution of predecessors, we consider the following: For any pair \((x,T)\), the probability that \( x \) has exactly \( i \) predecessors in \( T \) is given by \( \mathbf{P}_{n, m} \left( \frac{n_i}{n} \right) \rightarrow \frac{\alpha_i^*}{i!} \) as \( n \) becomes large. Also, as \( n \rightarrow \infty \),

\[ \mathbf{P}\{x = x\} \rightarrow 1/n \rightarrow 0. \] Hence, writing for convenience, \( \frac{\alpha_i^*}{i!} = \theta_i \) and

\[ \mathbf{P}\{x = j\} = \psi_j \], we have
(14.56) \[ \psi_j = \phi_0 \]
\[ \psi_2 = \phi_1 \psi_1 \]
\[ \vdots \]
\[ \psi_j = \sum_{l=1}^{k} c_l \sum_{r_1, r_2, \ldots, r_k} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_k} \]
\[ \sum_{t=1}^{j-1} r_t = j-1 \]

Consider the function \( y = h(x) = \sum_{t=1}^{\infty} x^t \psi_t \). Then

(14.57) \[ y = h(x) = x(\phi_0 + \phi_1 \sum_{r_1=1}^{\infty} x^{r_1} \psi_{r_1} + \phi_2 \sum_{r_2=1}^{\infty} \sum_{r_1=1}^{r_2} x^{r_1+r_2} \psi_{r_1} \psi_{r_2} + \ldots + \phi_k \sum_{r_k=1}^{\infty} \sum_{r_{k-1}=1}^{r_k} \sum_{r_1=1}^{r_{k-1}} \ldots \sum_{r_1=1}^{r_k} x^{r_1+r_2+\ldots+r_k} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_k}) \]
\[ = x(\phi_0 + \phi_1 h(x) + \phi_2 h^2(x) + \ldots + \phi_k h^k(x)) \]
\[ = x(\phi_0 + \phi_1 y + \phi_2 y^2 + \ldots + \phi_k y^k) \]
\[ = x g(y) \quad \text{say}. \]

Then

(14.58) \[ h(x) = \sum_{j=1}^{\infty} \frac{(h^{(j)}(x))}{j!} x^j = \sum_{j=1}^{\infty} \frac{\left( \frac{\partial^{(j-1)} g(y)}{\partial y^{j-1}} \right)_{y=0}}{j!} x^j \]

so that

(14.59) \[ \psi_j = \frac{1}{j!} \left( \frac{\partial^{(j-1)}}{\partial y^{j-1}} g(y) \right)_{y=0} = \frac{1}{j!} \left( \frac{\partial^{(j-1)}}{\partial y^{j-1}} (\phi_0 + \phi_1 y + \phi_2 y^2 + \ldots + \phi_k y^k) \right)_{y=0} \]
\[ = \frac{1}{j!} \sum_{r_0, r_1, \ldots, r_k} \frac{r_0 r_1 \ldots r_k}{r_0! r_1! \ldots r_k!} \sum_{l=0}^{\infty} \sum_{r_1, r_2, \ldots, r_k} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_k} \]
\[ \sum_{t=0}^{j} r_t = j \]
\[ \sum_{t=1}^{k} r_t = j-1 \]
\[
\sum_{r_0, r_1, \ldots, r_k} \frac{1}{r_0! r_1! \ldots r_k!} r_0 r_1 \ldots r_k \frac{1}{r_0! r_1! \ldots r_k!} \frac{1}{r_0! r_1! \ldots r_k!} \frac{1}{r_0! r_1! \ldots r_k!} \\
\sum_{t=0}^{k} \frac{r_t}{j} - 1 \\
\sum_{t=1}^{k} \frac{tr_t}{j} - 1 - \frac{1}{j}
\]

\[
\frac{(\beta^*-1)^j k! j}{\beta^*^{j+k+1} j} \frac{\beta^*^{k+1}}{(2\pi j)^{1/2}(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^{j+k+1}} \\
= \frac{1}{(2\pi j)^{1/2} \beta^*^{j+k+1} j} \frac{\beta^*^{k+1}}{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^{j+k+1}}
\]

for large \( j \).

When \( k = 2 \), \( \theta_0 = \frac{1}{2} (2-\sqrt{2}) \), \( \theta_1 = \frac{1}{2} (2-\sqrt{2}) \sqrt{2} \), \( \theta_2 = \frac{1}{2} (2-\sqrt{2}) \) and

\[
y = y(x) = \frac{1}{2} (2-\sqrt{2}) x (1+\sqrt{2}y^2, y^2). \quad \text{From this we have}
\]

\[
\psi_j \sim \frac{(\sqrt{2}-1)^j}{j^{1/2}} \sum_{r_2=0}^{j} \frac{1}{(r_2+1)! (j-2r_2)! r_2!} r_2 \\
= \frac{1}{(2\pi j)^{1/2} \beta^*^{j+k+1} j} \frac{\beta^*^{k+1}}{(k-(k-1)\beta^*)^{1/2}(\beta^*-1)^{j+k+1}}
\]

for large \( j \).
REFERENCE

Figure 1

Structure Classes in $J$ when $n = 5$, $n^2 = 3.125$

$m = 1$, $N_\phi = 1$

120

$m = 1$, $N_\phi = 2$

120

$m = 1$, $N_\phi = 3$

120

$m = 1$, $N_\phi = 4$

120

$m = 1$, $N_\phi = 5$

120
Figure 1 (Continued)

Structure Classes in J when $n = 5$, $n^2 = 3.125$

- $m = 2$, $N_\phi = 2$
  - $120$

- $m = 2$, $N_\phi = 1$
  - $120$

- $m = 2$, $N_\phi = 3$
  - $120$

- $m = 3$, $N_\phi = 3$
  - $120$

- $m = 3$, $N_\phi = 4$
  - $120$

- $m = 3$, $N_\phi = 5$
  - $120$

- $m = 3$, $N_\phi = 6$
  - $120$
Figure 1 (continued)

Structure Classes in \( J \) when \( n = 5; n^2 = 3.125 \)

- \( n = 3, \ N_\beta = 5 \)
  - \( \bullet \bullet \bullet \bullet \) 20

- \( n = 4, \ N_\beta = 4 \)
  - \( \bullet \bullet \bullet \bullet \) 20

- \( n = 4, \ N_\beta = 5 \)
  - \( \bullet \bullet \bullet \) 10

- \( n = 5, \ N_\beta = 5 \)
  - \( \bullet \bullet \bullet \bullet \bullet \) 1
## Table 1

### Probability of 1 Cycles in a Permutation of 1 Elements

<table>
<thead>
<tr>
<th>j</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000 0000</td>
<td>0.5000 0000</td>
<td>0.3333 3333</td>
<td>0.2500 0000</td>
</tr>
<tr>
<td>2</td>
<td>0.5000 0000</td>
<td>0.5000 0000</td>
<td>0.5000 0000</td>
<td>0.4583 3333</td>
</tr>
<tr>
<td>3</td>
<td>0.3333 3333</td>
<td>0.3333 3333</td>
<td>0.5000 0000</td>
<td>0.6666 6666</td>
</tr>
<tr>
<td>4</td>
<td>0.2500 0000</td>
<td>0.2500 0000</td>
<td>0.4583 3333</td>
<td>0.6666 6666</td>
</tr>
<tr>
<td>5</td>
<td>0.2000 0000</td>
<td>0.2000 0000</td>
<td>0.4583 3333</td>
<td>0.6666 6666</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1666 6666 6666</td>
<td>0.1666 6666 6666</td>
<td>0.1666 6666 6666</td>
<td>0.1666 6666 6666</td>
<td>0.1666 6666 6666</td>
</tr>
<tr>
<td>2</td>
<td>0.3805 5555 5555</td>
<td>0.3805 5555 5555</td>
<td>0.3805 5555 5555</td>
<td>0.3805 5555 5555</td>
<td>0.3805 5555 5555</td>
</tr>
<tr>
<td>3</td>
<td>0.7106 7106 7106</td>
<td>0.7106 7106 7106</td>
<td>0.7106 7106 7106</td>
<td>0.7106 7106 7106</td>
<td>0.7106 7106 7106</td>
</tr>
<tr>
<td>4</td>
<td>1.0000 0000 0000</td>
<td>1.0000 0000 0000</td>
<td>1.0000 0000 0000</td>
<td>1.0000 0000 0000</td>
<td>1.0000 0000 0000</td>
</tr>
<tr>
<td>5</td>
<td>0.0013 6666 6666</td>
<td>0.0013 6666 6666</td>
<td>0.0013 6666 6666</td>
<td>0.0013 6666 6666</td>
<td>0.0013 6666 6666</td>
</tr>
<tr>
<td>6</td>
<td>0.0001 9189 1891</td>
<td>0.0001 9189 1891</td>
<td>0.0001 9189 1891</td>
<td>0.0001 9189 1891</td>
<td>0.0001 9189 1891</td>
</tr>
<tr>
<td>7</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
</tr>
<tr>
<td>8</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
</tr>
<tr>
<td>9</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
</tr>
<tr>
<td>10</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
<td>0.0000 0000 0000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0909 0909</td>
<td>0.0833 3333</td>
<td>0.0769 5714</td>
<td>0.0714 2857</td>
<td>0.0666 6666</td>
</tr>
<tr>
<td>2</td>
<td>0.2667 5472</td>
<td>0.2500 0000</td>
<td>0.2373 0061</td>
<td>0.2272 7031</td>
<td>0.2166 6666</td>
</tr>
<tr>
<td>3</td>
<td>0.3805 5555</td>
<td>0.3636 3636</td>
<td>0.3488 3721</td>
<td>0.3359 7874</td>
<td>0.3243 2434</td>
</tr>
<tr>
<td>4</td>
<td>0.5000 0000</td>
<td>0.4818 1818</td>
<td>0.4642 8571</td>
<td>0.4482 7586</td>
<td>0.4337 2093</td>
</tr>
<tr>
<td>5</td>
<td>0.5992 5926</td>
<td>0.5882 3529</td>
<td>0.5783 1301</td>
<td>0.5690 4762</td>
<td>0.5603 7735</td>
</tr>
<tr>
<td>6</td>
<td>0.7727 2727</td>
<td>0.7619 0476</td>
<td>0.7518 8679</td>
<td>0.7424 2424</td>
<td>0.7335 2452</td>
</tr>
<tr>
<td>7</td>
<td>0.9167 9167</td>
<td>0.9056 6034</td>
<td>0.9011 2359</td>
<td>0.8967 5276</td>
<td>0.8928 5714</td>
</tr>
<tr>
<td>8</td>
<td>0.9900 9000</td>
<td>0.9787 2341</td>
<td>0.9725 8714</td>
<td>0.9664 0435</td>
<td>0.9605 6102</td>
</tr>
<tr>
<td>9</td>
<td>0.9980 3922</td>
<td>0.9841 2738</td>
<td>0.9790 2453</td>
<td>0.9736 2651</td>
<td>0.9682 5394</td>
</tr>
<tr>
<td>10</td>
<td>0.9999 9999</td>
<td>0.9929 5185</td>
<td>0.9878 7901</td>
<td>0.9828 5714</td>
<td>0.9778 2609</td>
</tr>
<tr>
<td>(i)</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>1</td>
<td>.06250</td>
<td>.05882</td>
<td>.05555</td>
<td>.05263</td>
<td>.05000</td>
</tr>
<tr>
<td>2</td>
<td>.20738</td>
<td>.19886</td>
<td>.19108</td>
<td>.18395</td>
<td>.17738</td>
</tr>
<tr>
<td>3</td>
<td>.29469</td>
<td>.28955</td>
<td>.28451</td>
<td>.27960</td>
<td>.27481</td>
</tr>
<tr>
<td>4</td>
<td>.31669</td>
<td>.31557</td>
<td>.31202</td>
<td>.30912</td>
<td>.30622</td>
</tr>
<tr>
<td>5</td>
<td>.12937</td>
<td>.12859</td>
<td>.12765</td>
<td>.12716</td>
<td>.12725</td>
</tr>
<tr>
<td>6</td>
<td>.04825</td>
<td>.04709</td>
<td>.04627</td>
<td>.04590</td>
<td>.04590</td>
</tr>
<tr>
<td>7</td>
<td>.01303</td>
<td>.01303</td>
<td>.01297</td>
<td>.01287</td>
<td>.01282</td>
</tr>
<tr>
<td>8</td>
<td>.00321</td>
<td>.00320</td>
<td>.00319</td>
<td>.00317</td>
<td>.00315</td>
</tr>
<tr>
<td>9</td>
<td>.00026</td>
<td>.00025</td>
<td>.00024</td>
<td>.00023</td>
<td>.00023</td>
</tr>
<tr>
<td>10</td>
<td>.00008</td>
<td>.00007</td>
<td>.00006</td>
<td>.00005</td>
<td>.00005</td>
</tr>
<tr>
<td>11</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>12</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>13</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>14</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>15</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>16</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>17</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>18</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>19</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>20</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(j)</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.04761</td>
<td>.04575</td>
<td>.04317</td>
<td>.04165</td>
<td>.04000</td>
</tr>
<tr>
<td>2</td>
<td>.17132</td>
<td>.16569</td>
<td>.16047</td>
<td>.15559</td>
<td>.15103</td>
</tr>
<tr>
<td>3</td>
<td>.27018</td>
<td>.26568</td>
<td>.26133</td>
<td>.25713</td>
<td>.25307</td>
</tr>
<tr>
<td>4</td>
<td>.25192</td>
<td>.25375</td>
<td>.25331</td>
<td>.25361</td>
<td>.25375</td>
</tr>
<tr>
<td>5</td>
<td>.15732</td>
<td>.15162</td>
<td>.14658</td>
<td>.14294</td>
<td>.14261</td>
</tr>
<tr>
<td>6</td>
<td>.07046</td>
<td>.06746</td>
<td>.06420</td>
<td>.06151</td>
<td>.06135</td>
</tr>
<tr>
<td>7</td>
<td>.02361</td>
<td>.02574</td>
<td>.02786</td>
<td>.02996</td>
<td>.03023</td>
</tr>
<tr>
<td>8</td>
<td>.00609</td>
<td>.00599</td>
<td>.00571</td>
<td>.00551</td>
<td>.00550</td>
</tr>
<tr>
<td>9</td>
<td>.00123</td>
<td>.00125</td>
<td>.00159</td>
<td>.00195</td>
<td>.00201</td>
</tr>
<tr>
<td>10</td>
<td>.00019</td>
<td>.00024</td>
<td>.00028</td>
<td>.00035</td>
<td>.00037</td>
</tr>
<tr>
<td>11</td>
<td>.00002</td>
<td>.00003</td>
<td>.00004</td>
<td>.00005</td>
<td>.00006</td>
</tr>
<tr>
<td>12</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>13</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>14</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>15</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>16</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>17</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>18</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>19</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>20</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>21</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>22</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>23</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>24</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>25</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
<td>.00000</td>
</tr>
</tbody>
</table>
Figure 2

Distribution of structure sizes

\[
P\{c = k\}
\]

Distribution of predecessors

\[
P\{p = k\}
\]

Distribution of number of structures

\[
P\{m = k\}
\]

Distribution of six-lengths

\[
P\{s = k\}
\]