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CONSTRUCTION AND SELECTION OF SMOOTHING FORMULAS

L. S. Dederick

Research and Development Project No. TB3-0110V

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

B A L L I S T I C R E S E A R C H L A B O R A T O R I E S

REPORT NO. 863

May 1953

CONSTRUCTION AND SELECTION OF SMOOTHING FORMULAS

L. S. Dederick

Project No. TB3-0110V of the Research and
Development Division, Ordnance Corps

A B E R D E E N P R O V I N G G R O U N D, M A R Y L A N D

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ABSTRACT

Many types of phenomena are investigated by being observed at a large number of successive points or successive time intervals. All observations are subject to error; and it frequently happens that the errors of successive observations are such that the inferred velocities, accelerations, or the like, are wholly incredible as a result of these errors of observation. In cases of this sort some process of smoothing is a necessity; and even where no absurdity is obvious a smoothing process may be highly desirable. This means a process by which each observed value is altered slightly so as to bring it into reasonable relations with those that precede and follow it. In this there are obviously two conflicting aims, (a) to achieve reasonable smoothness, and (b) to alter the observed data by reasonably small amounts. The purpose of this report is to present a method by which these two conflicting aims are more effectively reconciled than by any previously used method.

Construction and Selection of Smoothing Formulas

1. Nature of the Problem

The general problem of smoothing consists in the determination in the most plausible manner available of the values of a function believed to have some simple form of regularity or 'smoothness', when this regularity has been obscured by the fact that the values of the function which are supposed known have been determined by an imperfect method which may alter each one by some unknown error. These errors are assumed to have a random distribution. Their source need not be explicitly specified. They may arise as errors of observation, or as an accumulation of errors of computation such as rounding errors, errors due to omission of small terms, etc. A typical source consists in the accidental errors occurring in a sequence of runs on the Differential Analyzer or other analog machine. The values originally obtained by the imperfect method will be called the original or crude values, and the values by which we replace these as more plausible, the smoothed or adjusted values. The difference between a crude value and the corresponding adjusted value will be called a residual. We shall limit our consideration here to entries listed against only one argument, and shall further assume that the values of this argument are equally spaced, and hence may be taken without loss of generality as consecutive integers.

2. Tabular Differences

The criteria that we shall apply for the smoothness of a sequence will be expressed in terms of tabular differences. The notation to be used for these will be as follows. The original values will be called u_o, u_1, u_2, ----, where u_o may be either the first entry in the sequence or some entry at which we wish to begin consideration. Successive tabular differences will be written as follows:

$$\Delta_i' u = u_i - u_{i-1}$$

$$\Delta_i'' u = \Delta_i' u - \Delta_{i-1}' u$$

$$\Delta_i''' u = \Delta_i'' u - \Delta_{i-1}'' u$$

- - - - -

$$\Delta_i^{(j+1)} u = \Delta_i^{(j)} u - \Delta_{i-1}^{(j)} u$$

The tabular difference of any order is defined in terms of differences of the next lower order. It may therefore be ultimately expressed in terms of values of the function. Thus

$$\Delta_i'' u = u_i - 2u_{i-1} + u_{i-2}$$

$$\Delta_i''' u = u_i - 3u_{i-1} + 3u_{i-2} - u_{i-3}$$

and in general,

$$\Delta_i^{(j)} u = \sum_{k=0}^{j} b_k^j u_{i-k},$$

where the coefficients b_k^j are the binomial coefficients with alternating signs, namely

$$b_k^j = \frac{(-1)^k \, j!}{k! \, (j-k)!}$$

If the sequence u consists of the successive values of a polynomial of degree m, it is well known that its tabular differences of degree m + 1 are equal to zero. If the values of a sequence are in some interval approximately equal to the values of a polynomial, then the differences of some order may be expected to be small. We take then the criterion of smoothness that the tabular differences of some order shall be small; and the process of smoothing must tend to decrease on the whole the magnitudes of the tabular differences of some order. Practically all sequences that we shall deal with will have the following characteristics. The numerical values of the first order differences are smaller than those of the values of the function, the second order differences smaller than these, and so on for a certain number of orders. For any one of these orders the values fall into consecutive groups of several values each, having one algebraic sign within the group. For higher orders the numerical values increase with the order, and in any order, not more than two consecutive values keep the same sign. If we think of the order for which the numerical magnitudes are least as m + 1, then we may regard the sequence, at least in some limited neighborhood, as approximately represented by a polynomial of degree m. If the differences of order m + 1 are small, the sequence approximates closely to the values of a polynomial and may be regarded as smooth. If these differences are not small the process of smoothing is one that tends to diminish their magnitudes. If we take m as determined by the sequence in the manner just described, then any procedure for smoothing must be such that if applied to a polynomial of degree m it will leave its values unchanged.

3. Smoothing Methods

The use of a smoothing formula is only one of the possible methods of smoothing. By a smoothing formula is here meant a formula by which the adjusted value of any entry is obtained as an explicit function of the crude values of a limited number of entries in the neighborhood of the one in question. The use of such a formula may be contrasted with curve fitting in that the latter makes use of all the entries in a sequence, in adjusting any one, whereas a smoothing formula uses only those in a certain neighborhood. In curve fitting also a form of function must be selected in advance which has one or more parameters to be determined to produce the best fit to the given sequence. In smoothing by formula no such selection is necessary. Smoothing by formula may also be contrasted with smoothing 'by eye' in that the whole operation is summed up in one substitution in a formula for each entry, rather than consisting of successive operations each dependent on the results of the preceding ones.

4. Characters of the Formulas

Every smoothing formula considered will have the form of a linear homogeneous function of the values of the neighboring entries. These entries will be confined to those whose arguments differ from that of the entry to be adjusted by not more than a certain number n, which will be called the spread of the formula. Except for entries within n steps of the beginning or end of the sequence, the formula will then involve 2n+1 entries, the one to be smoothed and n entries on each side of it. With the exception noted, the entries other than the one to be smoothed occur in pairs, the two members of a pair having arguments equidistant from that of the entry to be smoothed. In all such symmetric cases the formulas will also be symmetric, that is they will have equal coefficients for the two entries of any such pair. If, however, the entry to be smoothed is within n steps of the beginning or end the formula can not be symmetric as a whole, and the symmetry of the coefficients does not apply. This case will for the present be excluded from consideration. A method for dealing with entries near the beginning or end of a sequence will be discussed in paragraph 18. In addition to the spread n we shall use another parameter m, called the order, and defined as follows. If a smoothing formula is applied to a sequence all of whose entries are equal, it is clear that the formula should reproduce these equal values. If a smoothing formula, when applied to the successive values of a polynomial of degree m, reproduces those values, but fails to reproduce the values of a polynomial of higher order, then the formula will be said to be of order m.

5. Principal Objects

The objects of this paper are to obtain for various given values of n and m those coefficients which shall make the formula in each case, a practical optimum, and to formulate a criterion for selecting a suitable value of m for the smoothing of any particular sequence. A criterion for n is much more difficult to obtain, since the value of n determines the amount of smoothing, and this may depend on whether a certain peculiarity in the crude values is regarded as merely an irregularity to be smoothed out or a genuine indication of a characteristic of the function. As regards the first problem it is necessary to specify what the criteria are for an optimum. In general it is desirable that the residuals shall remain small. This is the criterion of fidelity. It is even more desirable that in any neighborhood not too small they shall have a random distribution with a mean approximately equal to zero. This is a criterion for avoidance of bias. Obviously one important criterion of merit in a smoothing formula is the attaining of smoothness. More concretely this is the diminution of the numerical magnitudes of the tabular differences of some order. The amount of this diminution may be called the criterion of power. For selections of coefficients which are of substantially equal merit from the foregoing points of view, there may be differences in the ease of evaluation. This may be spoken of as the criterion of convenience. Of these four desiderata we may regard the avoidance of bias as the most important. Clearly a glaring fault in a smoothing formula would be that it should make a systematic shift in one direction of any considerable number of consecutive entries. It may be shown that the avoidance of this evil depends entirely on the selection of m. Fidelity and power are somewhat in conflict. For a given value of m, an increase in n will increase power but is likely to decrease fidelity. In fact for a given value of m we may proceed to one extreme by taking n = 0. This gives maximum fidelity since there is no smoothing and hence every residual is zero. On the other hand we may proceed to the other extreme by taking n large enough to include all the

entries in the sequence. The coefficients could then be chosen so as to replace all the entries by the values of a single polynomial. This would give perfect smoothness, and hence maximum power, but might produce large residuals and hence poor fidelity. For given values of m and n the principal problem is to select the coefficients to obtain maximum power. Selection on this basis does not appear to militate against fidelity. In general the criterion of convenience should come last. It may appear as a small modification of coefficients which have been determined as optimum from some form of the criteria that have been mentioned.

6. Conditions on the Coefficients to Secure a Specified Order

From the preceding we see that a smoothing formula of spread n may be written in the form

$$U_i = \sum_{j=-n}^{n} c_j u_{i+j}, \tag{1}$$

where u_i is the original value of an entry and U_i is the smoothed value. If we require the formula to be of order m this imposes certain conditions on the coefficients c_j. The definition of m requires that $U_i = u_i$ when the values of u_i are those of a polynomial of degree m. Let this polynomial be

$$u_{i+j} = \sum_{k=0}^{m} a_k j^k .$$

Then $u_i = a_o$, and

$$U_i = \sum_{j=-n}^{n} c_j \sum_{k=0}^{m} a_k j^k = u_i = a_o \tag{2}$$

In this sum the coefficient of a_k is $\sum_{j=-n}^{n} c_j j^k$. Since in (2) the coefficient of a_o is 1 and that of every other a_k is zero we have as the conditions for the order m that

$$\sum_{j=-n}^{n} c_j = 1 \tag{3}$$

and

$$\sum_{j=-n}^{n} c_j j^k = 0 \tag{4}$$

when $0 < k \leq m$. This makes m+1 conditions imposed upon 2n+1 coefficients since we are excluding entries within n steps of the beginning or end. Hence if $m > 2n$ there is no solution. If $m = 2n$ there is one solution which is readily seen to be $c_o = 1$, $c_j = 0$ when $j \neq 0$. This gives $U_i = u_i$ regardless of what values u_i has, and hence provides no smoothing. Therefore to obtain an actual smoothing formula we must have $2n > m$. We have noted that the coefficients are made symmetrical, that is $c_j = c_{-j}$. This results in (4) being satisfied automatically for every odd value of k. Thus formulas are always of odd order, at least as far as this specific condition is concerned. We shall subsequently derive further conditions which will distinguish between second and third order,

between fourth and fifth, etc. It should be understood, of course, that the coefficients in the formula do not vary with the argument of the entry being smoothed. Thus c_j does not vary with i. The conditions (3) and (4) imposed on the coefficients, cince they are linear relations, may be solved if desired for certain of the coefficients and the results substituted for these explicitly, leaving a smaller number of independent unknown coefficients to be determined by the application of other conditions.

7. Polynomial of Best Fit

We shall now discuss a method commonly used to complete the determination of the coefficients, and the reasons for not using this method. The method consists in finding coefficients which will give a smoothed value of the central entry considered which is equal to the value at this central point of that polynomial of degree m which is the best fit for the 2n+1 entries used. In speaking of the best fit we may understand this in either one of two ways. The usual way is to regard all 2n+1 entries as of equal weight. This involves the logical contradiction that in determining the smoothed value of an entry we restrict our attention to these 2n+1 entries, presumably because they are nearer than the others which we ignore, and yet we make no distinction among those we use, in spite of the fact that some are much nearer than others. On the other hand we may avoid this contradiction by giving the 2n+1 entries unequal weights, the greatest weight being assigned to the entry being smoothed, and the weights decreasing from this to relatively small weights assigned to the outermost entries. In this case it remains to determine the exact assignment of weights. For example the binomial coefficients might be regarded as a plausible selection, or some set of weights based on the probability function. There are other more desirable selections than these; but we shall prefer here to proceed upon a principle more explicitly related to what may be plausibly regarded as the main advantage of unequal weights. The error in the crude value of any one entry makes a contribution to the smoothed value of each of the 2n+1 entries which contain it. If the weights in the formula are equal, this contribution suddenly jumps from zero to a certain value, remains fixed at this value for 2n+1 entries, and then suddenly disappears. With unequal weights its contribution is first small, then rises gradually to a maximum and gradually decreases. Thus each error contributes in a relatively smooth manner to the final result, and we do not depend nearly so much for the final smoothness upon the errors balancing each other locally. Since, in addition to its logical superiority, a merit of unequal weights consists in attaining greater smoothness than that given by equal weights, it seems preferable to seek this greater smoothness more directly and explicitly. This will be done in the following; but a digression will be made here to discuss the significance of order.

8. Characteristics of the Order of a Smoothing Formula

If the graph of the smoothed function (assumed to give approximately the true values) is a curve, then the effect of smoothing by a formula of order one is in general that of fitting a straight line to a group of 2n+1 points distributed along the curve or near it, and taking the value at the central point on this line for the smoothed value. If the curve has no inflection point in the interval considered, this procedure has the effect of bringing the smoothed point to the concave side of the curve. The straight line, being chosen to give small residuals for all the 2n+1 points used (by minimizing

the sum of their squares), is likely to give residuals systematically of one sign near the middle of the interval, and of the other sign near the ends, these 'residuals' referring of course merely to the determination of the particular straight line and not to the smoothing as a whole. The general effect is to put all the smoothed points on the concave side of the true curve, and thus introduce a systematic bias over any interval in which the curve continues concave one way. If we use a formula of order two, we are attempting to fit a parabola to the points. This will avoid the obvious type of bias just mentioned, but may leave a bias not so easily obvious, that is some tendency for the residuals to remain entirely or predominantly of one sign over some considerable interval. This is an indication that the order used is still too small. We have indicated in Paragraph 2 a method for determining m so as to eliminate bias. If this value of m is used, m+1 conditions are imposed on the coefficients. The remaining coefficients may then be determined with a view to obtaining maximum smoothness. Since the order m+1 is the lowest for which the magnitudes of the differences are due mainly to the roughness rather than to the true values, the remaining degrees of freedom are used in an endeavor to diminish the magnitudes of the differences of order m+1. The method for this will now be discussed.

9. Further Conditions on the Coefficients

We shall now assume that the order of the formula used is sufficient to avoid appreciable bias. This means that the true values of the function can be represented with sufficient accuracy by a polynomial of degree m in the interval extending n steps on each side of the value in question. Now it is a property of a polynomial of degree m that the differences of order m+1 of its values at equal intervals shall be zero. Let us write

$$u_i = t_i + e_i$$

where t_i denotes the true value of the function and e_i the error. Then with sufficient accuracy we shall have

$$\Delta_i^{(m+1)} t = 0.$$

Let us express differences of all orders in terms of the values of the original function by means of the appropriate coefficients, as in Paragraph 2. Then

$$\Delta_i^{(m+1)} U = \sum_{k=0}^{m+1} b_k^{m+1} U_{i-k}$$

$$= \sum_{k=0}^{m+1} b_k^{m+1} \sum_{j=-n}^{n} c_j u_{i-k+j}$$

$$= \sum_{k=0}^{m+1} b_k^{m+1} \sum_{j=-n}^{n} c_j t_{i-k+j} + \sum_{k=0}^{m+1} b_k^{m+1} \sum_{j=-n}^{n} c_j e_{i-k+j}$$

The first of these sums is

$$\sum_{k=0}^{m+1} b_k^{m+1} \sum_{j=-n}^{n} c_j t_{i-k+j} = \sum_{j=-n}^{n} c_j \sum_{k=0}^{m+1} b_k^{m+1} t_{i-k+j} = \sum_{j=-n}^{n} c_j \Delta_{i+j}^{(m+1)} t,$$

which is equal to zero to the approximation that we have used. Consequently we have approximately

$$\Delta_i^{(m+1)} U = \sum_{k=0}^{m+1} b_k^{m+1} \sum_{j=-n}^{n} c_j e_{i-k+j}.$$

This is the expression which we desire to minimize in general for any particular values of m and n in order to achieve an optimum set of values of the coefficients c_j. It is a linear homogeneous form in the random variables e_{i-k+j}, and hence will in general be a minimum if the sum of the squares of the coefficients of these variables is minimized. Now a particular variable e_{i-r} has the coefficient

$$\sum_{k=0}^{m+1} b_k^{m+1} c_{k-r}$$

For the optimum selection of coefficients c_{k-r}, the sum of the squares of this expression must be minimized for all relevant values of r. Since r=k-j the extreme values of r will be -n and n+m+1. We shall thus obtain in general the smallest values of $\Delta_i^{(m+1)} U$ and hence the optimum coefficients c_j if we require that

$$\sum_{r=-n}^{n+m+1} \left[\sum_{k=0}^{m+1} b_k^{m+1} c_{k-r} \right]^2 = \text{a minimum} \qquad (5)$$

subject to the requirements (3) and (4) of paragraph 6 for making the formula one of order m. In equation (5) the summation indicated will give certain values of k-r outside the limits -n and n. For these there will be no values of the coefficient c_{k-r}. This will present no difficulty if we merely define the coefficients c_j as being zero whenever $j^2 > n^2$, and determine non zero values for the others. In this condition for a minimum the coefficients b_k^{m+1} are known, whereas the coefficients c_{k-r} are unknowns to be determined subject only to the conditions already imposed. These latter may be used to eliminate one or more of the unknowns and the indicated sum minimized as a function of all those that are left. It is, however, much more convenient and systematic not to perform these eliminations, but to set up criteria for satisfying all the conditions. It may be noted that no explicit criterion of fidelity

is introduced. It can be seen, however, that the procedure indicated tends to insure a measure of fidelity for the following reason. If m is chosen so as to eliminate bias, the smoothed values in any interval not too small will be bracketed by the crude values. This tends to make them lie as near to the crude values as is consistent with smoothness.

10. Formulation of the Conditions

The problem just formulated is that of minimizing a certain function, as given in (5), subject to the fulfilment of certain conditions namely those given in (3) and (4). This problem is a special case of a general one formulated as follows. It is required to minimize the function $\varphi(x_1, x_2, ---, x_r)$ subject to the s conditions given by the equations

$$F_i\ (x_1, --- x_r) = 0 \qquad i = 1, 2, --- s$$

where $s < r$. It is a well known theorem (cf. Goursat, Cours d'Analyse § 61.) and easy to prove that a necessary condition for this is the vanishing of all the (s+1) row determinants in the matrix

$$\left\| \begin{matrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & - - - - - & \frac{\partial F_1}{\partial x_r} \\ \frac{\partial F_2}{\partial x_1} & - - - - - - - & & - \\ & - - - - - - - - & & \\ \frac{\partial F_s}{\partial x_1} & - - - - - - - & & \frac{\partial F_s}{\partial x_r} \\ \frac{\partial \varphi}{\partial x_1} & - - - - - - - - & & \frac{\partial \varphi}{\partial x_r} \end{matrix} \right\|$$

For the cases considered here this condition may be replaced by the vanishing of those determinants whose columns are consecutive. For the particular problem the arguments $x_1, \ldots x_r$ become $c_{-n}, \ldots, c_n$. The partial derivatives are

$$\frac{\partial F_1}{\partial c_i} = 1, \quad \frac{\partial F_2}{\partial c_i} = i, \quad \frac{\partial F_3}{\partial c_i} = i^2, \quad \frac{\partial F_s}{\partial c_i} = i^m.$$

The function φ is that given in (5) namely

$$\varphi(c_{-n}, \ldots c_n) = \sum_{r=-n}^{n+m+1} \left[\sum_{k=0}^{m+1} b_k^{m+1}\ c_{k-r} \right]^2$$

From the symmetry assumed for the coefficients $c_{-n}, \ldots c_n$, c_{k-r} may be replaced by c_{r-k}. The bracket then becomes a tabular difference of order m+1 of the

sequence of coefficients $c_{-n}, \ldots, c_n$ and we may write

$$\varphi(c_{-n}, \ldots c_n) = \sum_{r=-n}^{n+m+1} \left(\Delta_r^{(m+1)}\right)^2$$

where $\Delta'_r, \Delta''_r, \ldots$ denote tabular differences of the coefficients $c_{-n}, \ldots c_n$. The last row in the matrix consists of the successive values of

$$\frac{\partial \varphi}{\partial c_i} = \sum_{r=-n}^{n+m+1} 2 \sum_{k=o}^{m+1} b_k^{m+1} c_{r-k} b_{r-i}^{m+1} .$$

This may be written with the more restricted range of summation

$$\frac{\partial \varphi}{\partial c_i} = 2 \sum_{r=1}^{i+m+1} \sum_{k=o}^{m+1} b_k^{m+1} b_{r-i}^{m+1} c_{r-k},$$

since outside this range $b_{r-i}^{m+1} = o$. To order this according to the values of c_{r-k}, let $s = r-k$. Then we have

$$\frac{\partial \varphi}{\partial c_i} = 2 \sum_{s=i-m-1}^{i+m+1} c_s \sum_{k=o}^{m+1} b_k^{m+1} b_{s-i+k}^{m+1} .$$

A well known theorem on binomial coefficients states that

$$\sum_{k=o}^{j} C_k^j C_{q+k}^p = C_{q+j}^{p+j},$$

and it is easy to see that a similar theorem except for a sign applies to the coefficients b that are used here. It follows then that

$$\frac{\partial \varphi}{\partial c_i} = 2 \sum_{s=i-m-1}^{i+m+1} (-1)^{m+1} b_{s-i+m+1}^{2m+2} c_s = 2(-1)^{m+1} \Delta_{i+m+1}^{(2m+2)}$$

The necessary condition for the required minimum therefore is the vanishing of all the $(m+2)$ row determinants of the matrix

$$\left\| \begin{matrix} 1 & -n & (-n)^2 \ldots & (-n)^m & \Delta_{-n+m+1}^{(2m+2)} \\ 1 & -n+1 & (-n+1)^2 \ldots & (-n+1)^m & \Delta_{-n+m+2}^{(2m+2)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & n & n^2 \ldots & n^m & \Delta_{n+m+1}^{(2m+2)} \end{matrix} \right\|$$

where the rows and columns have been interchanged merely for convenience of writing. Or we may write

$$\begin{vmatrix} 1 & i-2m-2 & (i-2m-2)^2 \cdots (i-2m-2)^m & \Delta_{i-m-1}^{(2m+2)} \\ 1 & i-2m-1 & (i-2m-1)^2 \cdots (i-2m-1)^m & \Delta_{i-m}^{(2m+2)} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & i-m-1 & (i-m-1)^2 \cdots (i-m-1)^m & \Delta_{i}^{(2m+2)} \end{vmatrix} = 0$$

with $i = -n+2m+2, -n+2m+3, \cdots, n+m+1$. To evaluate this we may replace the elements of the last row by that linear combination of this row and the preceding rows which will give in each column the difference of order $m+1$ in that column regarded as a sequence, then replace the elements in the next to the last row similarly by the differences of order m, and so on. This will give $\Delta_i^{(3m+3)}$ as the last element in the last row, a numerical constant in each of the remaining elements in the principal diagonal and a zero in each element below this diagonal. The necessary condition for a minimum then finally becomes $\Delta_i^{(3m+3)} = 0$ for $i=-n+2m+2, \cdots n+m+1$, where the differences are those of the coefficients c_i regarded as a tabular sequence, with the understanding that $c_i = 0$ if $i^2 > n^2$, and that $c_i = c_{-i}$. One very significant feature of this result consists in the range of values of i where it is applicable. The coefficient c_i is zero by definition if, $i^2 > n^2$, but the vanishing of $\Delta_i^{(3m+3)}$ occurs for at least one difference at each end of the sequence that involves one of these zero coefficients, in fact for a number of them at each end equal to $m+1$. This fact is basic in the technique of determining the numerical values of the coefficients.

11. <u>Case Where $m = 0$.</u>

For this case $\Delta_i''' = 0$ when $-n+2 \leq i \leq n+1$. This implies that c_i may be expressed as a quadratic in i, which from the symmetry of the coefficients may be written

$$c_i = a_1 i^2 + a_o$$

when $-n-1 \leq i \leq n+1$. From the vanishing of c_{n+1} we get at once

$$a_1 (n+1)^2 + a_o = 0 \quad \text{or}$$

$$c_i = a_1 \left[i^2 - (n+1)^2 \right] .$$

Thus the coefficients are determined except for the constant factor a_1. This may be determined by (3) namely the condition that the sum of the coefficients is equal to unity. In this and all similar cases it is sometimes practical to take the constant factor, here a_1, entirely at convenience, and to divide the results so obtained by the sum of all the coefficients. In this case we take $a_1 = -1$ and write

$$c_i = (n+1)^2 - i^2$$

with the understanding that the results of using the formula thus produced are always to be divided by the sum of the coefficients. Thus for $n = 1$ we have

$$U_o = (3u_{-1} + 4u_o + 3u_1) \div 10$$

For $n = 2$,

$$U_o = (5u_{-2} + 8u_{-1} + 9u_o + 8u_1 + 5u_2) \div 35$$

Using merely detached coefficients we may write for $n = 3$

$$(7 + 12 + 15 + 16 + 15 + 12 + 7) \div 84$$

For $n = 4$

$$(9 + 16 + 21 + 24 + 25 + 24 + 21 + 16 + 9) \div 165$$

For $n = 5$

$$(11 + 20 + 27 + 32 + 35 + 36 + 35 + 32 + 27 + 20 + 11) \div 286$$

and so on for any n.

12. Use of the Binomial Coefficients

Before proceeding to larger values of m, we may observe that from the analogy of the preceding we may expect to have to perform six summations or finite integrations for $m = 1$, nine for $m = 2$ and so on. Now altho we proceeded without difficulty in the simple case where $m = 0$ by expressing c_i as a polynomial in i, we shall find that finite differences and more particularly finite summations are handled much more easily if c_i and its successive differences are expressed as a sum of multiples of binomial coefficients instead of a sum of multiples of powers, i.e. if we deal with terms containing C_j^i rather than i^j. We shall use a definition slightly more general than the binomial coefficient proper, namely

$$C_j^i = \frac{i(i-1)----(i-j+1)}{j!}, \qquad (6)$$

where j is a positive integer but i may be <u>any</u> integer. Further let $C_o^i = 1$ for any i and $C_j^i = 0$ if $j<0$ for any i. From (6) we have

$$C_j^i = 0 \text{ if } 0 \leq i < j.$$

Also

$$\mathfrak{C}^{i}_{j} = (-1)^{j}\, C^{j-i-1}_{j} \tag{7}$$

From the definition we may readily derive

$$C^{i+1}_{j+1} - C^{i}_{j+1} = C^{i}_{j}. \tag{8}$$

If we call this $\Delta' C^{i+1}_{j+1}$, we can also write

$$\sum_{i=j}^{k} C^{i}_{j} = C^{k+1}_{j+1} \tag{9}$$

It is clear that (8) and (9) are vastly easier to use than the corresponding formulas for differences and sums of powers, particularly the latter which involve Bernoulli numbers or their equivalent. In any case, at each successive step of integration an unknown additive constant must be introduced. In half the cases this may be readily found as follows. From the symmetry of the coefficients, $c_{-i} = c_i$. Hence

$$\Delta'_{-i} = c_{-i} - c_{-i-1} = c_i - c_{i+1} = -\Delta'_{i+1}$$

Likewise

$$\Delta''_{-i} = \Delta''_{i+2},\ \Delta'''_{-i} = -\Delta'''_{i+3} \quad \text{and in general}$$

$$\Delta^{(j)}_{-i} = (-1)^{j}\, \Delta^{(j)}_{i+j} \tag{10}$$

This combined with (7) gives an equation involving the constant of integration. In half the cases this determines its value. In the other half it is merely an identity, and the constant must be found otherwise.

There is one feature of the binomial coefficients which is not so simple as that for powers. If the smoothing coefficients c_i are expressed as a polynomial, their symmetry which makes $c_i = c_{-i}$ is provided by merely omitting the odd powers. With binomial coefficients we may proceed as follows. The essential feature is that the symmetry reduces the number of parameters to determine. This may be accomplished by joining two binomial coefficients in a fixed pattern. Let a difference in c_i of any order be

$$\Delta^{(j)}_{i} = a\, \mathfrak{C}^{i+p}_{q} + \mathfrak{C}^{i+p}_{q-1},$$

or a sum of multiples of such quantities. Then

$$\Delta^{(j)}_{j-i} = a\, C^{j-i+p}_{q} + C^{j-i+p}_{q-1}$$

By (7) this becomes

$$\Delta^{(j)}_{j-1} = (-1)^q(a C_q^{i-j-p+q-1} - C_{q-1}^{i-j-p+q-2})$$

$$= (-1)^q\left[a(C_q^{i-j-p+q-2} + C_{q-1}^{i-j-p+q-2}) - C_{q-1}^{i-j-p+q-2}\right]$$

$$= (-1)^q\left[a C_q^{i-j-p+q-2} + (a-1)\, C_{q-1}^{i-j-p+q-2}\right]$$

But by (10)

$$\Delta^{(j)}_{j-1} = (-1)^j\ \Delta^{(j)}_i = (-1)^j\ (a C_q^{i+p} + C_{q-1}^{i+p})$$

These two expressions for $\Delta^{(j)}_{j-1}$ will be equal if $a = 2$ and $2p = q-j-2$.

The terms which make up $\Delta^{(j)}_i$ will then have the form

$$2C_q^{i+p} + C_{q-1}^{i+p} \quad \text{with } 2p = q-j-2$$

13. <u>Case Where m = 1.</u> We start here with $\Delta^{vi}_i = 0$ when $-n+4 \leqslant i \leqslant n+2$. From this Δ^{v}_i is a constant; but the constant is zero by (10), so that $\Delta^{v}_i = 0$ when $-n+3 \leqslant i \leqslant n+2$. Thus the highest order non-vanishing difference is Δ^{iv}_i. Since this is a constant, writing it in the form just derived requires that $q = 0$. Hence $p = -3$ and

$$\Delta^{iv}_i = 2a_2\, C_o^{i-3} \quad \text{when } -n+2 \leqslant i \leqslant n+2$$

Then in succession

$$\Delta^{iii}_i = a_2(2C_1^{i-2} + C_o^{i-2}) \quad \text{when } -n+1 \leqslant i \leqslant n+2$$

$$\Delta^{ii}_i = a_2(2C_2^{i-1} + C_1^{i-1}) + 2a_1\, C_o^{i-2} \quad \text{when } -n \leqslant i \leqslant n+2$$

$$\Delta^{i}_i = a_2(2C_3^{i} + C_2^{i}) + a_1(2C_1^{i-1} + C_o^{i-1}) \quad \text{when } -n-1 \leqslant i \leqslant n+2$$

$$a_i = a_2(2C_4^{i+1} + C_3^{i+1}) + a_1\ (2C_2^{i} + C_1^{i}) + 2a_o\, C_o^{i-1} \quad \text{when } -n-2 \leqslant i \leqslant n+2$$

14. <u>General Procedure.</u> We shall now seek a procedure applicable to any value of m. The general expression for any order of difference of the

coefficients $c_{-n} \cdots c_n$ is

$$\Delta_i^{(j)} = \sum_k a_k \left(C_{2k-j-1}^{i+k-j-1} + 2C_{2k-j}^{i+k-j-1}\right), \qquad (11)$$

holding when $-n+j-m-1 \leq i \leq n+m+1$. The summation extends from $k = 0$ to an upper limit depending on m, the limit being given by the relation $2k \leq 3m+2$ This may be seen by noting that the highest order difference which may be different from zero is $j = 3m+2$. The second term in the parenthesis in (11) then has the lower index $2k-3m-2$ and this must be zero to make this term a constant. Of course terms with smaller values of k are introduced by the successive integrations.

Putting $j = 0$ gives

$$c_i = \sum_k a_k \left(C_{2k-1}^{i+k-1} + 2C_{2k}^{i+k-1}\right) \qquad (12)$$

holding when $-n-m-1 \leq i \leq n + m + 1$. Summing once more with respect to i gives

$$\sigma_i = \sum_k a_k \left(C_{2k}^{i+k} + 2C_{2k+1}^{i+k}\right). \qquad (13)$$

The meaning of σ_i is the sum of the coefficients beginning at some fixed point in the sequence and ending with c_i. To determine the fixed point we may observe that $c_o = 2a_o$ but $\sigma_o = a_o$. Hence the sum begins with <u>half</u> of the middle coefficient. This makes the sum of all the coefficients equal to $2\sigma_n$.

15. <u>Determination of the Additive Constants</u>. In order to evaluate the expression for c_i it remains to determine the values of $a_o, a_1 \ldots$, the number of these depending on m. When $m = 0$, there are only the two, a_o and a_1, and two conditions are needed to determine them. For larger values of m there are more constants and correspondingly more conditions. For all values of m we have the condition $2\sigma_n = 1$. The expression for c_i holds up to $i = n+m+1$, but $c_i = 0$ if $i > n$. Hence we have the conditions $c_{n+1} = 0$, $c_{n+2} = 0$, .. $c_{n+m+1} = 0$. Further conditions arising from equation (4) remain to be formulated, but enter for the first time only when $m=2$. In the following no upper limit will be indicated for k in the sums used, so that the equations written may be used for any value of m by extending the values of k suitably. On the other hand elimination will occur of certain terms with small values of k, or these terms may be written separately, and the notation $\sum_o$, $\sum_1$, $\sum_2$ will be used for sums in which, 0, 1, 2, ... respectively is the smallest value of k. In all such sums also the letter a without subscript will be used for a_k. Although the two-term expression for $\Delta_i^{(j)}$ is well adapted for the summation process, it will usually be

convenient for algebraic purposes to make use of the transformation

$$C^p_q + 2\,C^p_{q+1} = \frac{2p-q+1}{q+1}\;C^p_q\,. \tag{14}$$

Of course this is not applicable when $q = -1$. We have then

$$c_i = 2a_o + \sum_1^i \frac{a_i}{k}\,C^{i+k-1}_{2k-1} \tag{15}$$

$$\sigma_i = (2i+1)\sum_o \frac{a}{2k+1}\,C^{i+k}_{2k} \tag{16}$$

For the actual evaluation of the coefficients c_i the following is perhaps the most convenient form.

$$c_i = 2a_o + i^2\left[a_1 + \frac{i^2-1}{3.4}\left\{a_2 + \frac{i^2-4}{5.6}\left[a_3 + \frac{i^2-9}{7.8}\left\{a_4 + \cdots\right\}\right]\right\}\right]$$

For particular values of i this becomes

$$\begin{aligned}
c_o &= 2a_o \\
c_1 = c_1 &= 2a_o + a_1 \\
c_2 = c_{-2} &= 2a_o + 4a_1 + a_2 \\
c_3 = c_{-3} &= 2a_o + 9a_1 + 6a_2 + a_3 \\
c_4 = c_{-4} &= 2a_o + 16a_1 + 20a_2 + 8a_3 + a_4 \\
c_5 = c_{-5} &= 2a_o + 25a_1 + 50a_2 + 35a_, + 10a_4 + a_5 \\
c_6 = c_{-6} &= 2a_o + 36a_1 + 105a_2 + 112a_3 + 54a_4 + 12a_5 + a_6 \\
c_7 = c_{-7} &= 2a_o + 49a_1 + 196a_2 + 294a_3 + 210a_4 + 77a_5 + 14a_6 + a_7 \\
c_8 = c_{-8} &= 2a_o + 64a_1 + 336a_2 + 672a_3 + 660a_4 + 352a_5 + 104a_6 + 16a_7 + \\
c_9 = c_{-9} &= 2a_o + 81a_1 + 540a_2 + 1386a_3 + 1782a_4 + 1287a_5 + 546a_6 + 135a_7 + \\
c_{10} = c_{-10} &= 2a_o + 100a_1 + 825a_2 + 2640a_3 + 4290a_4 + 4004a_5 + 2275a_6 + 800a_7 + \\
c_{11} = c_{-11} &= 2a_o + 121a_1 + 1210a_2 + 4719a_3 + 9438a_4 + 11011a_5 + 8008a_6 + 3740a_7 + \\
c_{12} = c_{-12} &= 2a_o + 144a_1 + 1716a_2 + 8008a_3 + 19305a_4 + 27456a_5 + 24752a_6 + 14688a_7 +
\end{aligned}$$

with the understanding that the formulas stop with a_1 when $m = 0$, with a_2 when $m = 1$, with a_4 when $m = 2$, with a_5 when $m = 3$, with a_7 when $m = 4$, etc. It remains to determine the values of $a_o, a_1, \ldots$ for various values of m and n.

16. <u>Details for Various Values of m.</u> For any value of m we have the two equations $2\sigma_n = 1$ and $c_{n+1} = 0$. From these we may write

$$P_o = \frac{c_n}{2n+1} = \frac{1}{2(2n+1)} = \sum_o \frac{a}{2k+1} C_{2k}^{n+k} = a_o + \frac{a_1 n(n+1)}{2.3} + \sum_2 \frac{a}{2k+1} C_{2k}^{n+k} ;$$

$$P_1 = \frac{c_{n+1}}{n+1} = 0 = \frac{2a_o}{n+1} + a_1(n+1) + \sum_2 \frac{a}{k} C_{2k-1}^{n+k} ;$$

$$Q_1 = \frac{nP_1 - 6P_o}{2n+3} = \frac{-3}{(2n+1)(2n+3)} = \frac{-2a_o}{n+1} + \frac{a_2 n(n+1)(n+2)}{2^2 \cdot 3.5} + \sum_3 \frac{a(k-1)}{k(2k+1)} C_{2k-1}^{n+k} ,$$

$$Q_2 = \frac{(n+1)P_1 - 2P_o}{2n+3} = \frac{-1}{(2n+1)(2n+3)} = \frac{a_1(n+1)}{3} + \frac{a_2 n(n+1)(n+2)}{2.3.5} + \sum_3 \frac{a}{2k+1} C_{2k-1}^{n+k} .$$

When $m = 0$, $k < 2$, and hence

$$a_o = \frac{3(n+1)}{2(2n+1)(2n+3)}, \qquad a_1 = \frac{-3}{(n+1)(2n+1)(2n+3)}$$

When $m > 0$, we have the additional condition

$c_{n+2} = 0$, or

$$P_2 = \frac{c_{n+2}}{n+2} = 0 = \frac{2a_o}{n+2} + a_1(n+2) + \sum_2 \frac{a}{k} C_{2k-1}^{n+k+1}$$

$$Q_3 = \frac{(n+1)P_2 - (n+2)P_1}{2n+3} = 0 = \frac{-2a_o}{(n+1)(n+2)} + \frac{a_2(n+1)(n+2)}{2^2 \cdot 3} + \sum_3 \frac{a(k-1)}{k(2k-1)} C_{2k-2}^{n+k}$$

$$Q_4 = \frac{(n+2)P_2 - (n+1)P_1}{2n+3} = 0 = a_1 + \frac{a_2(n+1)(n+2)}{2 \cdot 3} + \sum_3 \frac{a}{2k-1} C_{2k-2}^{n+k}$$

Before proceeding further it is convenient to introduce a notation for certain types of products.

Let $\quad q(x,y) = x(x+1)(x+2)\ldots(x+y-1),$

$\qquad r(x,y) = x(x+2)(x+4)\ldots(x+2y-2)$

Thus $q(x,y)$ and $r(x,y)$ each denote a produot of y factors of which the first is x, the successive factors increasing by 1 in the case of q, and by 2 in the case of r. With this notation we continue.

$$R_1 = \frac{nQ_3 - 5Q_1}{2n+5} = \frac{3.5}{r(2n+1,3)} = \frac{2^2 a_o}{q(n+1,2)} + \frac{a_3 q(n,4)}{2^2 . 3^2 . 5.7} + \sum_4 \frac{a q(k-2,2)}{kr(2k-1,2)} C_{2k-2}^{n+k}$$

$$R_2 = \frac{nQ_4 - 5Q_2}{2n+5} = \frac{5}{r(2n+1,3)} = \frac{-1}{3} + \frac{a_3 q(n,4)}{2^3 . 3.5.7} + \sum_4 \frac{a(k-2)}{r(2k-1,2)} C_{2k-2}^{n+k}$$

$$R_3 = \frac{(n+2)Q_3 - Q_1}{2n+5} = \frac{(n+1)Q_4 - 3Q_2}{2n+5} = \frac{3}{r(2n+1,3)} = \frac{a_2 q(n+1,2)}{2.3.5} + \frac{a_3 q(n,4)}{2^2.3.5.7} + \sum_4 \frac{a(k-1)}{r(2k-1,2)} C_{2k-2}^{n+k}.$$

When $m = 1$, $k < 3$, and hence

$$a_o = \frac{3.5q(n+1,2)}{2^2 r(2n+1,3)}, \qquad a_1 = \frac{-3.5}{r(2n+1,3)}, \qquad a_2 = \frac{2.3^2.5}{q(n+1,2)\, r(2n+1,3)}$$

When $m > 1$, we have two additional conditions,

$c_{n+3} = 0$ and $\sigma' = \sum_{i=1}^{n} i^2 c_i = 0$. The former gives us, in the same manner as before,

$$P_3 = \frac{c_{n+3}}{n+3} = 0 = \frac{2a_o}{n+3} + a_1(n+3) + \sum_2 \frac{a}{k} C_{2k-1}^{n+k+2}$$

In order to utilize the summation properties of the binomial coefficients we make the following transformation in evaluating σ'

$$i^2 c_i = \sum_k a(2i^2 C_{2k}^{i+k-1} + i^2 C_{2k-1}^{i+k-1}).$$

Now

$$i^2 C_{2k}^{i+k-1} = (i+k)(i+k-1)C_{2k}^{i+k-1} + (i+k)C_{2k}^{i+k-1} + k^2 C_{2k}^{i+k-1}$$

$$= (2k+2)(2k+1)\, C_{2k+2}^{i+k} + (2k+1)\, C_{2k+1}^{i+k} + k^2 C_{2k}^{i+k-1}$$

Likewise

$$i^2 C_{2k-1}^{i+k-1} = 2k(2k+1)C_{2k+1}^{i+k} + k^2 C_{2k-1}^{i+k-1}$$

Hence

$$i^2 c_i = \sum_k a\left[4(k+1)(2k+1)C_{2k+2}^{i+k} + 2(k+1)(2k+1)C_{2k+1}^{i+k} + 2k^2 C_{2k}^{i+k-1} + k^2 C_{2k-1}^{i+k-1}\right.$$

$$= \sum_k a\left[2(k+1)(2k+1)(2C_{2k+2}^{i+k} + C_{2k+1}^{i+k}) + k^2(2C_{2k}^{i+k-1} + C_{2k-1}^{i+k-1})\right]$$

and

$$\sigma' = \sum_k a\left[2(k+1)(2k+1)(2C_{2k+3}^{n+k+1} + C_{2k+2}^{n+k+1}) + k^2(2C_{2k+1}^{n+k} + C_{2k}^{n+k})\right]$$

$$= \sum_k a \left[\frac{2(k+1)(2k+1)(2n+1)}{2k+3} C_{2k+2}^{n+k+1} + \frac{k^2(2n+1)}{2k+1} C_{2k}^{n+k} \right]$$

$$= (2n+1) \sum_k a \left[\frac{(n+k+1)(n-k)}{2k+3} + \frac{k^2}{2k+1} \right] C_{2k}^{n+k}$$

$$= (2n+1) \sum_k a \frac{n(n+1)(2k+1)-k}{(2k+1)(2k+3)} C_{2k}^{n+k} = 0.$$

$$P' = \frac{\sigma'}{n(n+1)(2n+1)} = 0 = \frac{a_o}{3} + a_1 \frac{3n(n+1)-1}{2.3.5} + \sum_2 a \left[\frac{1}{2k+3} - \frac{k}{q(n,2)r(2k+1,2)} \right] C_{2k}^{n+k}$$

$$Q_5 = \frac{(n+2)P_3-(n+3)P_2}{2n+5} = 0 = \frac{-2\,a_o}{q(n+2,2)} + \frac{a_2 q(n+2,2)}{2^2.3} + \sum_3 \frac{a(k-1)}{k(2k-1)} C_{2k-2}^{n+k+1}$$

$$Q_6 = \frac{(n+3)P_3-(n+2)P_2}{2n+5} = 0 = a_1 + \frac{a_2 q(n+2,2)}{2.3} + \sum_3 \frac{a}{2k-1} C_{2k-2}^{n+k+1}$$

$$R_4 = \frac{(n+1)Q_5-(n+3)Q_3}{n+2} = 0 = \frac{2^3 a_o}{q(n+1,3)} + \frac{a_3 q(n+1,3)}{2.3^2.5} + \sum_4 \frac{a(k-2)}{k(2k-1)} C_{2k-3}^{n+k}$$

$$R_5 = \frac{(n+1)Q_6-(n+3)Q_4}{n+2} = 0 = \frac{-2\,a_1}{n+2} + \frac{a_3 q(n+1,3)}{2^2.3.5} + \sum_4 \frac{a(k-2)}{(k-1)(2k-1)} C_{2k-3}^{n+k}$$

$$R_6 = \frac{(n+3)Q_5-(n+1)Q_3}{n+2} = Q_6 - Q_4 = 0 = \frac{a_2(n+2)}{3} + \frac{a_2 q(n+1,3)}{2.3.5} + \sum_4 \frac{a}{2k-1} C_{2k-3}^{n+k}$$

$$S_1 = \frac{nR_4-14R_1}{2n+7} = \frac{-2.3.5.7}{r(2n+1,4)} = \frac{-2^3.3\,a_o}{q(n+1,3)} + \frac{a_4 q(n,5)}{2^4.3^3.5.7} + \sum_5 \frac{a\,q(k-3,2)}{kr(2k-1,2)} C_{2k-3}^{n+k}$$

$$S_2 = \frac{nR_5-14R_2}{2n+7} = \frac{-2.5.7}{r(2n+1,4)} = \frac{2^2 a_1}{3(n+2)} + \frac{a_4 q(n,5)}{2^2.3^4.5.7} + \sum_5 \frac{a\,q(k-3,2)}{(k-1)r(2k-1,2)} C_{2k-3}^{n+k}$$

$$S_3 = \frac{nR_6-14R_3}{2n+7} = \frac{-2.3.7}{r(2n+1,4)} = \frac{-a_2(n+2)}{3.5} + \frac{a_4 q(n,5)}{2^3.3^3.5.7} + \sum_5 \frac{a(k-3)}{r(2k-1,2)} C_{2k-3}^{n+k}$$

$$S_4 = \frac{(n+3)R_4-2R_1}{2n+7} = \frac{(n+2)R_5-6R_2}{2n+7} = \frac{(n+1)R_6-10R_3}{2n+7}$$

$$= \frac{-2.3.5}{r(2n+1,4)} = \frac{a_3 q(n+1,3)}{2.3.5.7} + \frac{a_4 q(n,5)}{2^2.3^3.5.7} + \sum_5 \frac{a(k-2)}{r(2k-1,2)} C_{2k-3}^{n+k}$$

$$Q_7 = \frac{5n(n+1)P' - (3n^2+3n-1)P_o}{(2n-1)(2n+3)} = \frac{-(3n^2+3n-1)}{2r(2n-1,3)} = \frac{-a_o}{3} + \frac{a_2 q(n-1,4)}{2^3.3.5.7} + \sum_3 \frac{a(k-1)}{r(2k+1,2)} C_{2k}^{n+k}$$

$$Q_8 = \frac{n(n+1)(3P'-P_o)}{(2n-1)(2n+3)} = \frac{-n(n+1)}{2r(2n-1,3)} = \frac{a_1 q(n,2)}{2.3.5} + \frac{a_2 q(n-1,4)}{2^2.3.5.7} + \sum_3 \frac{a\,k}{r(2k+1,2)} C_{2k}^{n+k}$$

$$R_7 = \frac{(n-1)Q_1 - 14Q_7}{2n+5} = \frac{5(3n^2+6n-2)}{r(2n-1,4)} = \frac{4a_o}{3(n+1)} + \frac{a_3 q(n-1,5)}{2^2.3^4.5.7} + \sum_4 \frac{a\,q(k-2,2)}{kr(2k+1,2)} C_{2k-1}^{n+k}$$

$$R_8 = \frac{(n-1)Q_2 - 14Q_8}{2n+5} = \frac{5n^2+10n-1}{r(2n-1,4)} = \frac{-a_1(n+1)}{3.5} + \frac{a_3 q(n-1,5)}{2^3 3^3.5.7} + \sum_4 \frac{a(k-2)}{r(2k+1,2)} C_{2k-1}^{n+k}$$

$$R_9 = \frac{(n+1)Q_1 - 6Q_7}{(2n+5)} = \frac{nQ_2 - 10(n^2+n)Q_8}{2n+5} = \frac{3n(n+2)}{r(2n-1,4)} = \frac{a_2 q(n,3)}{2.3.5.7} + \frac{a_3 q(n-1,5)}{2^2.3^3.5.7} + \sum_4 \frac{a(k-1)}{r(2k+1,2)} C_{2k-1}^{n+k}$$

$$S_5 = \frac{(n-1)R_1 - 9R_7}{2n+7} = \frac{-3.5.7(n^2+3n-1)}{r(2n-1,5)} = \frac{-4a_o}{(q(n+1,2)} + \frac{a_4 q(n-1,6)}{2^5.3^3.5.7.11} + \sum_5 \frac{aq(k-3,3)}{kr(2k-1,3)} C_{2k-2}^{n+k}$$

$$S_6 = \frac{(n-1)R_2 - 9R_8}{2n+7} = \frac{-7(5n^2+15n-2)}{r(2n-1,5)} = \frac{2a_1}{3.5} + \frac{a_4 q(n-1,6)}{2^3.3^4.5.7.11} + \sum_5 \frac{a\,q(k-3,2)}{r(2k-1,3)} C_{2k-2}^{n+k}$$

$$S_7 = \frac{(n-1)R_3 - 9R_9}{2n+7} = \frac{-3(7n^2+21n-1)}{r(2n-1,5)} = \frac{-a_2 q(n+1,2)}{2.3.5.7} + \frac{a_4 q(n-1,6)}{2^4.3^3.5.7.11} + \sum_5 \frac{a\,r(k-3,2)}{r(2k-1,3)} C_{2k-2}^{n+k}$$

$$S_8 = \frac{(n+2)R_1 - 3R_7}{2n+7} = \frac{(n+1)R_2 - 5R_8}{2n+7} = \frac{nR_3 - 7R_9}{2n+7}$$

$$= \frac{-3.5n(n+3)}{r(2n-1,5)} = \frac{a_3 q(n,4)}{2^2.3^3.5.7} + \frac{a_4 q(n-1,6)}{2^3.3^3.5.7.11} + \sum_5 \frac{aq(k-2,2)}{r(2k-1,3)} C_{2k-2}^{n+k}$$

$$T_1 = \frac{(n-1)S_1 - 22S_5}{2n+9} = \frac{2.3^2.5.7(3n^2+12n-4)}{r(2n-1,6)} = \frac{2^5 a_0}{q(n+1,3)} + \sum_5 \frac{aq(k-4,3)}{kr(2k-1,3)} C_{2k-3}^{n+k}$$

$$T_2 = \frac{(n-1)S_2 - 22S_6}{2n+9} = \frac{2.3^2.7(5n^2+20n-3)}{r(2n-1,6)} = \frac{-2^2 a_1}{5(n+2)} + \sum_5 \frac{aq(k-4,3)}{(k-1)r(2k-1,3)} C_{2k-3}^{n+k}$$

$$T_3 = \frac{(n-1)S_3 - 22S_7}{2n+9} = \frac{2.3^3(7n^2+28n-2)}{r(2n-1,6)} = \frac{2a_2(n+2)}{3.5.7} + \sum_5 \frac{aq(k-4,2)}{r(2k-1,3)} C_{2k-3}^{n+k}$$

$$T_4 = \frac{(n-1)S_4 - 22S_8}{2n+9} = \frac{2.3.5(9n^2+36n-1)}{r(2n-1,6)} = \frac{-a_3 q(n+1,3)}{2.3^3.5.7} + \sum_5 \frac{ar(k-4,2)}{r(2k-1,3)} C_{2k-3}^{n+k}$$

$$T_5 = \frac{(n+3)S_1 - 6S_5}{2n+9} = \frac{(n+2)S_2 - 10S_6}{2n+9} = \frac{(n+1)S_3 - 14S_7}{2n+9} = \frac{nS_4 - 18S_8}{2n+9}$$

$$= \frac{2.3.5.7n(n+4)}{r(2n-1,6)} = \frac{a_4 q(n,5)}{2^2.3^3.5.7.11} + \sum_5 \frac{aq(k-3,2)}{r(2k-1,3)} C_{2k-3}^{n+k}$$

When $m = 2$, $k < 5$, and hence

$$a_0 = \frac{3^2.5.7(3n^2+12n-4)q(n+1,3)}{2^4 r(2n-1,6)}$$

$$a_1 = \frac{-3^2.5.7(5n^2+20n-3)(n+2)}{2r(2n-1,6)}$$

$$a_2 = \frac{3^4.5.7(7n^2+28n-2)}{(n+2)r(2n-1,6)}$$

$$a_3 = \frac{-2^2.3^4.5^2 7(9n^2+36n-1)}{q(n+1,3)r(2n-1,6)}$$

$$a_4 = \frac{2^3.3^4.5^2.7^2.11\,n(n+4)}{q(n,5)r(2n-1,6)}$$

By exactly similar methods one may obtain the results for $m = 3$ and $m = 4$. The algebraic details are rather tedious and will be omitted here. The results are as follows. For $m = 3$,

$$a_0 = \frac{3^2.7.11\,q(n+1,4)(3n^2 + 15n-5)}{2^4 r(2n-1,7)}$$

$$a_1 = \frac{-3^2.5.7.11\,q(n+2,2)(5n^2 + 25n-4)}{2^3 r(2n-1,7)}$$

$$a_2 = \frac{3^3.5.7.11(7n^2+35n-3)}{r(2n-1,7)}$$

$$a_3 = \frac{-2.3^4.5^2.7.11(9n^2 + 45n-2)}{q(n+2,2)r(2n-1,7)}$$

$$a_4 = \frac{2^3.3^4.5^2.7^2.11(11n^2 + 55n-1)}{q(n+1,4)r(2n-1,7)}$$

$$a_5 = \frac{-2^4.3^6.5^2.7^2.11.13}{q(n+1,4)r(2n-1,7)}$$

For $m = 4$,

$$a_0 = \frac{3^2.5.11.13\,q(n+1,5)(15n^4+180n^3+365n^2-1050n+252)}{2^6 r(2n-3,10)}$$

$$a_1 = \frac{-3^2.5.7.11.13\,q(n+2,3)(35n^4+420n^3+931n^2-1974n+180)}{2^5 r(2n-3,10)}$$

$$a_2 = \frac{3^5.5.7.11.13(n+3)(21n^4+252n^3+579n^2-1062n+40)}{2^3 r(2n-3,10)}$$

$$a_3 = \frac{-3^4.5^3.7.11.13(99n^4+1188n^3+2783n^2-4686n+72)}{2^2(n+3)r(2n-3,10)}$$

$$a_4 = \frac{2.3^4.5^3.7^2.11.13(143n^4+1716n^3+4069n^2-6474n+36)}{q(n+2,3)r(2n-3,10)}$$

$$a_5 = \frac{-2^2.3^8.5^3.7^2.11.13(65n^4+780n^3+1865n^2-2850n+4)}{q(n+1,5)r(2n-3,10)}$$

$$a_6 = \frac{2^4.3^7.5^3.7^2.11^2.13.17(15n^2+90n-107)}{q(n+1,5)r(2n-3,10)}$$

$$a_7 = \frac{-2^5.3^7.5^3.7^2.11^2.13^2.17.19}{q(n+1,5)r(2n-3,10)}$$

17. Tabulation of Coefficients. The values just given and those previously obtained, when substituted in (15) or in the enumerated special cases which follow (16) give the numerical values of c_i for $i^2 \leq n^2$, $n \leq 12$, and $m \leq 4$. They furnish smoothing formulas of degree up to the fourth and spread up to 25 entries ($n = 12$). They are tabulated in two forms, Tables 1 to 5 labeled as "Data Multipliers (Exact Values)" and Tables 6 - 10 labeled "Data Multipliers (Decimal Approximations)". Since these last are approximations it is not to be expected that they will satisfy exactly all the theoretical conditions. The value of c_o, however, has been adjusted so that the coefficients in all cases satisfy condition (3) namely the requirement that their sum be unity. This may be regarded as the most fundamental requirement. The values are given to 9 decimals. This is a convenient number for the large electronic machines. It means that for all practical purposes they are exact. If, however, it is desired to use the coefficients with ordinary desk calculators, the use of so many decimals is unduly laborious, and mere rounding to a small number of decimals will in general destroy the conformity to the theoretical conditions. Since the bias thus introduced could conceivably be significant it is recommended that if these tables are reduced to a small number of decimals the procedure be as follows. For $m = 0$ or $m = 1$, equation (3) is the only condition necessary to consider. In these cases we may take the ordinary rounded values of $c_1 \ldots c_n$ and find the central coefficient as

$$c_o = 1-2 \sum_{i=1}^{n} c_i \tag{17}$$

For $m = 2$ or $m = 3$ we must also satisfy equation (4) with $k = 2$. We take then the rounded values of $c_2 \ldots c_n$, write

$$c_1 = -\sum_{i=2}^{n} i^2 c_i \tag{18}$$

and find c_o from (17). For $m = 4$ we must also satisfy equation (4) with $k = 4$. If in this equation we substitute $k = 2$ and $k = 4$ and eliminate c_1 we get

$$c_2 = -\frac{1}{12} \sum_{i=3}^{n} i^2(i^2-1)c_i \tag{19}$$

In this case we take the rounded values of $c_3 \ldots c_n$ and substitute successively in (19), (18), and (17). In equations (17) and (18) the coefficients are obviously integers. In (19) they are also integers since $i^2(i^2-1)$ is devisible by 12 for any integer value of i. Hence the substitution in these equations involves no rounding and the results satisfy them exactly. In order to keep the changes in the central coefficients as small as possible it may be feasible to exercise some judgement in the rounding of the outer ones. For example for $m = 4$, $n = 4$, straight ~~rounding~~

rounding to four decimals would give $c_0 = .4976$, $c_1 = .3046$, $c_2 = -.0063$, $c_3 = -.0677$, and $c_4 = .0206$. If we accept these values of c_3 and c_4 and compute the others we get $c_2 = -.0058$, $c_1 = .3029$, and $c_0 = .5000$. On the other hand if we use $c_3 = -.0676$, we get $c_2 = -.0064$, $c_1 = .3044$, and $c_0 = .4980$, which are much nearer to the exact values. The reason is as follows. The six-decimal values are $c_3 = -.067688$ and $c_4 = .020624$. In the first case the rounding of c_3 and c_4 gives respective alterations of $-.000012$ and $-.000024$. In (15) these are multiplied by the coefficients -6 and -20 and give an alteration of .000552. In the second case the alteration in c_3 is made $+ .000088$. Now although this is larger than before it gives an alteration in c_2 of only $-.000048$. A little juggling of this sort may diminish appreciably the alterations in the central coefficients, which may become considerable for the larger values of n. The technique is to select the rounded value of c_n first and select in succession the rounding of c_{n-1}, c_{n-2}... so as to prevent the building up of a large alteration in c_2.

18. Entries Near the Beginning or End. We have noted that a formula of spread n cannot be applied within n steps of the beginning or end of a sequence because of the absence of some of the terms called for. It is theoretically possible to derive special asymmtrical formulas to cover these cases. They require, however, greatly increased labor both in derivation and in use. The following procedure appears adequate to deal with the values near the ends. Whatever method may be used for selecting n for the body of the table, it need not be regarded as inviolable. For a value near enough to either end to call for an entry outside the table with the regular value of n, we shall simply use a value of n enough diminished to avoid this difficulty. In general we may continue to use the regular value of m. However, we may not diminish n so far as to make $2n \leq m$, since then there will be no smoothing. When this occurs, m may be decreased also, being taken equal to $2n-1$. When this is done, each entry of the sequence is smoothed except the end one, where the original value is left unchanged. There are many tables in which the leading entry is definitely fixed (often at zero). The method just described is best adapted to these cases. If an end entry is not fixed in this way, the following method may be better. Let n be diminished as the end is approached as long as $2n > m$. The remaining values may then be filled in by taking as constant the tabular difference of order m.

19. Comparison with Standard Methods - Numerical Example. In Table 11 is given a set of observed values at quarter second intervals of so-called V angles in the flight of a certain projectile. These are angles in degrees at one station to be combined with a similar set from another station to give the orientation of the projectile axis. The table also gives tabular differences up to the fifth order. The next six tables (12-17) give the results of smoothing these values by six different methods, each likewise accompanied by five orders of tabular differences, and by the residuals. The first three of these tables make use of a spread of 11 values (n=5), and the next three of a spread of 25 values (n=12). In each of these

groups of three the first and second are based on a standard formula of the type in common use and referred to in Paragraph 7 as undesirable. In Tables 12 and 15 the degree is 3; in Tables 13 and 16 the degree is 5. The final table in each group (14 and 17) is based on the methods developed in this report with the degree m=3. The relative characteristics of the several smoothing methods are summarized in Table 18. To indicate relative fidelity standard deviation is recorded for each method, namely the root mean square value of all the residuals. To indicate the smoothness obtained, the root mean square value of each order of difference is recorded for the original values and for each method of smoothing.

In order to make an unbiased comparison of the different methods the same number of values for any particular kind of quantity is included in the mean. Thus there are 73 observed values here recorded; but applying an eleven point smoothing formula to these gives 63 smoothed values, and a 25 pt. formula 49 values. Accordingly only 49 values are included in the residuals, 48 in the first differences, 47 in the second, and so on.

Let us first compare the power of the standard formulas for different orders and different spreads, that is the measure of smoothness obtained as indicated by the diminution in the mean values of the tabular differences. In no case is the first order difference greatly diminished. For all higher orders, however, the diminution is very notable. Comparison of the different values shows that the smoothing power for the same spread is greater for the third order than for the fifth, but for the same order it is greater for a 25 point spread than for an 11 point spread, that this difference is always substantial and for the higher order differences quite noticable. Thus increasing the spread increases the power, but increasing the order diminishes it.

If we turn our attention now to fidelity and look at the residuals, we see an entirely different situation. For the same spread the fifth order formula gives greater fidelity (i.e. smaller residuals) than the third; but for the same order the 11 point spread gives better fidelity than the 25 point.

We can combine these remarks as applied to the standard method by saying that increasing the spread gives more smoothing but less fidelity, whereas increasing the order gives more fidelity but less smoothing. This presents something of a dilemma.

If now we turn to the results obtained by the method developed in this report, and compare them with the preceding, we tend to avoid this dilemma. The columns marked "Weighted" give the results of the new method for m=3 and two values of n. It will be seen that the change to the new method diminishes both the residuals and the higher order differences. In fact the residuals are diminished almost as much by the change to the new method with m still 3 as they are by changing to m=5 with the standard method. The first and second differences are diminished little if any; but the third and higher differences have their means decreased to a small fraction of the old results, the fraction varying from one-half down to one-fifth. When it is recalled that this improvement in smoothing power is accompanied by a substantial improvement in fidelity also, it would appear that the new formulas should replace those commonly used in all cases where the use of a smoothing formula is the appropriate procedure.

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Table 1

Data Multipliers

(Exact Values)

m = 0

i \ n	1	2	3	4	5	6	7	8	9	10	11	
0	4	9	16	25	36	49	64	81	100	121	144	169
±1	3	8	15	24	35	48	63	80	99	120	143	168
±2	0	5	12	21	32	45	60	77	96	117	140	165
±3		0	7	16	27	40	55	72	91	112	135	160
±4			0	9	20	33	48	65	84	105	128	153
±5				0	11	24	39	56	75	96	119	144
±6					0	13	28	45	64	85	108	133
±7						0	15	32	51	72	95	120
±8							0	17	36	57	80	105
±9								0	19	40	63	88
±10									0	21	44	69
±11										0	23	48
±12											0	25
Divisor	10	35	84	165	286	455	680	969	1330	1771	2300	2925

Table 2
Data Multipliers
(Exact Values)
m = 1

i \ n	1	2	3	4	5	6	7	8	9	10	11	12
0	3	12	100	75	147	784	432	675	3025	1452	2028	8281
±1	2	10	90	70	140	756	420	660	2970	1430	2002	8190
±2	0	5	63	56	120	675	385	616	2808	1365	1925	7920
±3	0	0	28	36	90	550	330	546	2548	1260	1800	7480
±4		0	0	15	55	396	260	455	2205	1120	1632	6885
±5			0	0	22	234	182	350	1800	952	1428	6156
±6				0	0	91	105	240	1360	765	1197	5320
±7					0	0	40	136	918	570	950	4410
±8						0	0	51	513	380	700	3465
±9							0	0	190	210	462	2530
±10								0	0	77	253	1656
±11									0	0	92	900
±12										0	0	325
Divisor	7	42	462	429	1001	6188	3876	6783	33649	17710	26910	118755

Table 3
Data Multipliers
(Exact Values)
m = 2

i \ n	2	3	4	5	6	7	8	9	10	11	12
0	160	885	805	25676	4032	40860	386595	209935	8506784	580853	538265
±1	84	630	648	22050	3600	37422	360360	198198	8108100	557700	519792
±2	-21	126	288	13050	2475	28182	288288	165438	6981975	491700	466752
±3	0	-126	-24	3300	1100	16016	188552	118482	5325600	392700	385968
±4	0	0	-99	-2475	0	4641	85995	67032	3427200	275400	287793
±5	0	0	0	-2574	-468	-2730	5040	21420	1610784	156978	184680
±6		0	0	0	-325	-4732	-38080	-9996	169575	54150	89376
±7			0	0	0	-2652	-41616	-23256	-701100	-19950	12936
±8				0	0	0	-20349	-20349	-957600	-58575	-37191
±9					0	0	0	-9044	-731500	-63250	-58696
±10						0	0	0	-302841	-44022	-55200
±11							0	0	0	-17250	-35880
±12								0	0	0	-13455
DIVISOR	286	2145	2431	92378	16796	193154	2042975	1225785	54367170	4032015	4032015

Table 4
Data Multipliers
(Exact Values)
m = 3

i \ n	2	3	4	5	6	7	8	9	10	11	12
0	111	469	2884	7308	48636	82764	334323	7627477	12739727	1856127	11489296
±1	56	324	2268	6160	42768	74844	308308	7135128	12046320	1769768	11027016
±2	-14	54	918	3410	27918	54054	238238	5783778	10115820	1526668	9714276
±3	0	-60	-132	660	10868	28028	145236	3913728	7360320	1171368	7759752
±4	0	0	-297	-715	-1287	5733	56056	1979208	4351320	767448	5468067
±5	0	0	0	-572	-5148	-6552	-6664	411264	1674432	383724	3182652
±6	0	0	0	0	-2860	-8092	-32844	-515508	-222870	79002	1220142
±7		0	0	0	0	-3672	-28424	-767448	-1149120	-111188	-191268
±8			0	0	0	0	-11305	-549423	-1213245	-182413	-956340
±9				0	0	0	0	-198968	-769120	-160908	-1128380
±10					0	0	0	0	-259578	-93610	-888030
±11						0	0	0	0	-29900	-484380
±12							0	0	0	0	-148005
Divisor	195	1105	8398	25194	193154	371450	1671525	42010995	76608285	12096045	80640300

Table 5
Data Multipliers
(Exact Values)
m = 4

i \ n	3	4	5	6	7	8	9	10	11	12
0	5744	51756	216012	239844	4093584	28450851	1518803	9739301	140056384	1105880464
±1	2475	31680	154440	187200	3378375	24393600	1338480	8760960	127992150	1023264000
±2	-990	-660	34320	72540	1706250	14432880	879840	6205680	95830020	799425000
±3	165	-7040	-30745	-16640	123305	3758720	344760	3035520	54041130	497988480
±4	0	2145	-15015	-32565	-567840	-2813160	-55080	348840	15348960	200016135
±5	0	0	9009	-4992	-380562	-3792768	-209304	-1116288	-10264617	-23836032
±6	0	0	0	8840	30940	-1447040	-155040	-1279080	-19205010	-136422660
±7	0	0	0	0	125970	620160	-29070	-656640	-14925735	-143542080
±8	0	0	0	0	0	712215	41895	19665	-5282640	-85842900
±9		0	0	0	0	0	30590	279680	2093575	-16596800
±10			0	0	0	0	0	157320	3848130	24454980
±11				0	0	0	0	0	1816425	27820800
±12					0	0	0	0	0	11555775
DIVISOR	9044	104006	520030	668610	12926460	100180065	5892945	41250615	642641160	5462449860

Table 6

Data Multipliers

Decimal Approximations

m = 0

i \ n	1	2	3	4	5	6	7	8	9	10	11	12
0	0.400000000	0.257142856	0.190476190	0.151515152	0.125874126	0.107692308	0.094117646	0.083591330	0.075187970	0.068322980	0.062608696	0.057777778
±1	0.300000000	0.228571429	0.178571429	0.145454545	0.122377623	0.105494504	0.092647059	0.082559340	0.074436090	0.067758328	0.062173913	0.057435898
±2		0.142857143	0.142857143	0.127272727	0.111888112	0.098901099	0.088235294	0.079463364	0.072180451	0.066064370	0.060869565	0.056410256
±3			0.083333333	0.096969697	0.094405594	0.087912088	0.080882353	0.074303406	0.068421053	0.063241107	0.058695652	0.054700855
±4				0.054545455	0.069930070	0.072527473	0.070588235	0.067079463	0.063157895	0.059288538	0.055652174	0.052307692
±5					0.038461538	0.052747253	0.057352941	0.057791538	0.056390977	0.054206663	0.051739130	0.049230769
±6						0.028571429	0.041176471	0.046439628	0.048120301	0.047995483	0.046956522	0.045470085
±7							0.022058824	0.033023736	0.038345865	0.040654997	0.041304348	0.041025641
±8								0.017543860	0.027067669	0.032185206	0.034782609	0.035897436
±9									0.014285714	0.022586110	0.027391304	0.030085470
±10										0.011857708	0.019130435	0.023589744
±11											0.010000000	0.016410256
±12												0.008547009

Table 7

Data Multipliers

Decimal Approximations

$m = 1$

i \ n	1	2	3	4	5	6	7	8	9	10	11	12
0	0.428571428	0.285714286	0.216450216	0.174825174	0.146853146	0.126696832	0.111455108	0.099513490	0.089898660	0.081987578	0.075362320	0.069731800
±1	0.285714286	0.238095238	0.194805195	0.163170163	0.139860140	0.122171946	0.108359134	0.097302080	0.088264139	0.080745341	0.074396136	0.068965517
±2		0.119047619	0.136363636	0.130536131	0.119880120	0.109082094	0.099329206	0.090815273	0.083449731	0.077075099	0.071534745	0.066691929
±3			0.060606061	0.083916084	0.089910090	0.088881707	0.085139319	0.080495356	0.075722904	0.071146245	0.066889632	0.062986822
±4				0.034965035	0.054945055	0.063994829	0.067079463	0.067079463	0.065529436	0.063241107	0.060646600	0.057976506
±5					0.021978022	0.037815126	0.046955624	0.051599587	0.053493417	0.053754941	0.053065775	0.051837817
±6						0.014705882	0.027089783	0.035382574	0.040417249	0.043195934	0.044461605	0.044798114
±7							0.010319917	0.020050125	0.027281643	0.032185206	0.035302861	0.037135279
±8								0.007518797	0.015245624	0.021456804	0.026012635	0.029177719
±9									0.005646527	0.011857708	0.017168339	0.021304366
±10										0.004347826	0.009401709	0.013944676
±11											0.003418803	0.007578628
±12												0.002736727

Table 8

Data Multipliers

Decimal Approximations

m = 2

i \ n	2	3	4	5	6	7	8	9	10	11	12
0	0.559440558	0.412587412	0.331139448	0.277944966	0.240057156	0.211541050	0.189231390	0.171265760	0.156469134	0.144060228	0.133497768
±1	0.293706294	0.293706294	0.266556972	0.238693195	0.214336747	0.193741782	0.176389823	0.161690672	0.149135958	0.138317938	0.128916187
±2	-0.073426573	0.058741259	0.118469766	0.141267401	0.147356513	0.145904304	0.141111859	0.134964941	0.128422631	0.121948951	0.115761474
±3		-0.058741259	-0.009872480	0.035722791	0.065491784	0.082918293	0.092292857	0.096658060	0.097956175	0.097395471	0.095725834
±4			-0.040723982	-0.026792093	0.000000000	0.024027460	0.042093026	0.054684957	0.063038043	0.068303317	0.071376967
±5				-0.027863777	-0.027863777	-0.014133800	0.002466991	0.017474516	0.029627880	0.038932891	0.045803401
±6					-0.019349845	-0.024498587	-0.018639484	-0.008154774	0.003119070	0.013430010	0.022166584
±7						-0.013729977	-0.020370293	-0.018972332	-0.012895650	-0.004947898	0.003208321
±8							-0.009960474	-0.016600791	-0.017613571	-0.014527476	-0.009223924
±9								-0.007378129	-0.013454811	-0.015686946	-0.014557486
±10									-0.005570292	-0.010918114	-0.013690425
±11										-0.004278258	-0.008898776
±12											-0.003337041

Table 9

Data Multipliers

Decimal Approximations

m = 3

	1	2	3	4	5	6	7	8	9	10	11	12
0	1,000000000	0.569230770	0.424434390	0.343415098	0.290069064	0.251799084	0.222813298	0.200010768	0.181559066	0.166296986	0.153449082	0.142475860
±1		0.287179487	0.293212670	0.270064301	0.244502659	0.221419178	0.201491452	0.184447137	0.169839539	0.157245656	0.146309640	0.136743241
±2		-0.071794872	0.048868778	0.109311741	0.135349686	0.144537519	0.145521605	0.142527333	0.137672959	0.132046031	0.126212163	0.120464284
±3			-0.054298643	-0.015718028	0.026196714	0.056265985	0.075455647	0.086888321	0.093159612	0.096077337	0.096838925	0.096226725
±4				-0.035365563	-0.028379773	-0.006663077	0.015434110	0.033535843	0.047111667	0.056799601	0.063446193	0.067808118
±5					-0.022703818	-0.026652309	-0.017638982	-0.003986779	0.009789437	0.021857062	0.031723096	0.039467264
±6						-0.014806838	-0.021784897	-0.019649123	-0.012270788	-0.002909216	0.006531226	0.015130673
±7							-0.009885584	-0.017004831	-0.018267789	-0.014999944	-0.009192095	-0.002371866
±8								-0.006763285	-0.013078076	-0.015836995	-0.015080384	-0.011859331
±9									-0.004736094	-0.010039645	-0.013302530	-0.013992755
±10										-0.003388380	-0.007738893	-0.011012236
±11											-0.002471882	-0.006006674
±12												-0.001835373

Table 10

Data Multipliers

Decimal Approximations

m = 4

i \ n	3	4	5	6	7	8	9	10	11	12
0	0.635117206	0.497625138	0.415383728	0.358720330	0.316682528	0.283997132	0.257732424	0.236100748	0.217938708	0.202451372
± 1	0.273662096	0.304597812	0.296982865	0.279983847	0.261353456	0.243497546	0.227132613	0.212383743	0.199165813	0.187326938
± 2	-0.109464839	-0.006345788	0.065996193	0.108493741	0.131996695	0.144069381	0.149303956	0.150438484	0.149119020	0.146349170
± 3	0.018244140	-0.067688403	-0.059121589	-0.024887453	0.009538961	0.037519640	0.058503855	0.073587267	0.084092233	0.091165776
± 4		0.020623810	-0.028873334	-0.048705523	-0.043928500	-0.028081036	-0.009346770	0.008456601	0.023884184	0.036616562
± 5			0.017324001	-0.007466236	-0.029440543	-0.037859508	-0.035517725	-0.027061124	-0.015972548	-0.004363616
± 6				0.013221459	0.002393540	-0.014444391	-0.026309426	-0.031007538	-0.029884500	-0.024974629
± 7					0.009745127	0.006190453	-0.004933017	-0.015918308	-0.023225613	-0.026277968
± 8						0.007109349	0.007109349	0.000476720	-0.008220202	-0.015715092
± 9							0.005190953	0.006780020	0.003257767	-0.003038344
± 10								0.003813761	0.005987992	0.004476925
± 11									0.002826500	0.005093099
± 12										0.002115493

TABLE 11
OBSERVED VALUES

ITEM NUMBER	V-Angle (Degrees)	DIFFERENCES				
		Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1					
2	13.0	-1.1				
3	12.6	-0.4	0.7			
4	12.0	-0.6	-0.2	-0.9		
5	10.8	-1.2	-0.6	-0.4	0.5	
6	9.7	-1.1	0.1	0.7	1.1	0.6
7	8.0	-1.7	-0.6	-0.7	-1.4	-2.5
8	7.7	-0.3	1.4	2.0	2.7	4.1
9	7.6	-0.1	0.2	-1.2	-3.2	-5.9
10	7.1	-0.5	-0.4	-0.6	0.6	3.8
11	7.1	0.0	0.5	0.9	1.5	0.9
12	7.6	0.5	0.5	0.0	-0.9	-2.4
13	7.5	-0.1	-0.6	-1.1	-1.1	-0.2
14	8.3	0.8	0.9	1.5	2.6	3.7
15	9.0	0.7	-0.1	-1.0	-2.5	-5.1
16	10.7	1.7	1.0	1.1	2.1	4.6
17	11.1	0.4	-1.3	-2.3	-3.4	-5.5
18	11.8	0.7	0.3	1.6	3.9	7.3
19	13.6	1.8	1.1	0.8	-0.8	-4.7
20	13.5	-0.1	-1.9	-3.0	-3.8	-3.0
21	14.8	1.3	1.4	3.3	6.3	10.1
22	14.4	-0.4	-1.7	-3.1	-6.4	-12.7
23	15.3	0.9	1.3	3.0	6.1	12.5
24	15.6	0.3	-0.6	-1.9	-4.9	-11.0
25	15.2	-0.4	-0.7	-0.1	1.8	6.7
26	14.2	-1.0	-0.6	0.1	0.2	-1.6
27	13.9	-0.3	0.7	1.3	1.2	1.0
28	14.1	0.2	0.5	-0.2	-1.5	-2.7
29	13.2	-0.9	-1.1	-1.6	-1.4	0.1
30	12.0	-1.2	-0.3	0.8	2.4	3.8
31	10.2	-1.8	-0.6	-0.3	-1.1	-3.5
32	9.6	-0.6	1.2	1.8	2.1	3.2
33	8.2	-1.4	-0.8	-2.0	-3.8	-5.9
34	6.6	-1.6	-0.2	0.6	2.6	6.4
35	5.1	-1.5	0.1	0.3	-0.3	-2.9
36	3.9	-1.2	0.3	0.2	-0.1	0.2
37	3.5	-0.4	0.8	0.5	0.3	0.4
38	3.2	-0.3	0.1	-0.7	-1.2	-1.5
39	3.2	0.0	0.3	0.2	0.9	2.1
40	2.9	-0.3	-0.3	-0.6	-0.8	-1.7
41	2.1	-0.8	-0.5	-0.2	0.4	1.2
42	2.2	0.1	0.9	1.4	1.6	1.2

TABLE 11
OBSERVED VALUES

ITEM NUMBER	V-Angle (degrees)	Differences				
		Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
43	2.6	0.4	0.3	-0.6	-2.0	-3.6
44	3.3	0.7	0.3	0.0	0.6	2.6
45	4.2	0.9	0.2	-0.1	-0.1	-0.7
46	5.4	1.2	0.3	0.1	0.2	0.3
47	7.6	2.2	1.0	0.7	0.6	0.4
48	8.5	0.9	-1.3	-2.3	-3.0	-3.6
49	10.5	2.0	1.1	2.4	4.7	7.7
50	10.5	0.0	-2.0	-3.1	-5.5	-10.2
51	11.5	1.0	1.0	3.0	6.1	11.6
52	12.7	1.2	0.2	-0.8	-3.8	-9.9
53	13.2	0.5	-0.7	-0.9	-0.1	3.7
54	13.9	0.7	0.2	0.9	1.8	1.9
55	13.7	-0.2	-0.9	-1.1	-2.0	-3.8
56	13.2	-0.5	-0.3	0.6	1.7	3.7
57	12.7	-0.5	0.0	0.3	-0.3	-2.0
58	11.1	-1.6	-1.1	-1.1	-1.4	-1.1
59	10.4	-0.7	-0.9	2.0	3.1	4.5
60	8.9	-1.5	-0.8	-1.7	-3.7	-6.8
61	7.5	-1.4	0.1	0.9	2.6	6.3
62	6.0	-1.5	-0.1	-0.2	-1.1	-3.7
63	3.9	-2.1	-0.6	-0.5	-0.3	0.8
64	3.1	-0.8	1.3	1.9	2.4	2.7
65	2.0	-1.1	-0.3	-1.6	-3.5	-5.9
66	1.1	-0.9	0.2	0.5	2.1	5.6
67	1.3	0.2	1.1	0.9	0.4	-1.7
68	-1.1	-2.4	-2.6	-3.7	-4.6	-5.0
69	-2.0	-0.9	1.5	4.1	7.8	12.4
70	-1.1	0.9	1.8	0.3	-3.8	-11.6
71	-1.1	0.0	-0.9	-2.7	-3.0	0.8
72	-0.4	0.7	0.7	1.6	4.3	7.3
73	2.1	2.5	1.8	1.1	-0.5	-4.8

TABLE 12
ADJUSTED VALUES
(STANDARD METHOD - 3rd. DEGREE - 11 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2.	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7	9.33	0.37					
7	8.0	8.67	-0.67	-0.66				
8	7.7	7.91	-0.21	-0.76	-0.10			
9	7.6	7.36	0.24	-0.55	0.21	0.31		
10	7.1	7.12	-0.02	-0.24	0.31	0.10	-0.21	
11	7.1	7.07	0.03	-0.05	0.19	-0.12	-0.22	-0.01
12	7.6	7.40	0.20	0.33	0.38	0.19	0.31	0.53
13	7.5	7.84	-0.34	0.44	0.11	-0.27	-0.46	-0.77
14	8.3	8.38	-0.08	0.54	0.10	-0.01	0.26	0.72
15	9.0	9.27	-0.27	0.89	0.35	0.25	0.26	0.00
16	10.7	10.16	0.54	0.89	0.00	-0.35	-0.60	-0.86
17	11.1	11.20	-0.10	1.04	0.15	0.15	0.50	1.10
18	11.8	12.18	-0.38	0.98	-0.06	-0.21	-0.36	-0.86
19	13.6	13.02	0.58	0.84	-0.14	-0.08	0.13	0.49
20	13.5	13.80	-0.30	0.78	-0.06	0.08	0.16	0.03
21	14.8	14.45	0.35	0.65	-0.13	-0.07	-0.15	-0.31
22	14.4	14.96	-0.56	0.51	-0.14	-0.01	0.06	0.21
23	15.3	15.12	0.18	0.16	-0.35	-0.21	-0.20	-0.26
24	15.6	15.08	0.52	-0.04	-0.20	0.15	0.36	0.56
25	15.2	15.05	0.15	-0.03	0.01	0.21	0.06	-0.30
26	14.2	14.80	-0.60	-0.25	-0.22	-0.23	-0.44	-0.50
27	13.9	14.34	-0.44	-0.46	-0.21	0.01	0.24	0.68
28	14.1	13.61	0.49	-0.73	-0.27	-0.06	-0.07	-0.31
29	13.2	12.79	0.41	-0.82	-0.09	0.18	0.24	0.31
30	12.0	11.88	0.12	-0.91	-0.09	0.00	-0.18	-0.42
31	10.2	10.79	-0.59	-1.09	-0.18	-0.09	-0.09	0.09
32	9.6	9.38	0.22	-1.41	-0.32	-0.14	-0.05	0.04
33	8.2	7.86	0.34	-1.52	-0.11	0.21	0.35	0.40
34	6.6	6.49	0.11	-1.37	0.15	0.26	0.05	-0.30
35	5.1	5.34	-0.24	-1.15	0.22	0.07	-0.19	-0.24
36	3.9	4.47	-0.57	-0.87	0.28	0.06	-0.01	0.18
37	3.5	3.65	-0.15	-0.82	0.05	-0.23	-0.29	-0.28
38	3.2	3.09	0.11	-0.56	0.26	0.21	0.44	0.73
39	3.2	2.72	0.48	-0.37	0.19	-0.07	-0.28	-0.72
40	2.9	2.50	0.40	-0.22	0.15	-0.04	0.03	0.31
41	2.1	2.41	-0.31	-0.09	0.13	-0.02	0.02	-0.01
42	2.2	2.38	-0.18	-0.03	0.06	-0.07	-0.05	-0.07
43	2.6	2.72	-0.12	0.34	0.37	0.31	0.38	0.43
44	3.3	3.32	-0.02	0.60	0.26	-0.11	-0.42	-0.80
45	4.2	4.45	-0.25	1.13	0.53	0.27	0.38	0.80

TABLE 12
ADJUSTED VALUES

(STANDARD METHOD - 3rd. DEGREE - 11 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
46	5.4	5.80	-0.40	1.35	0.22	-0.31	-0.58	-0.96
47	7.6	7.12	0.48	1.32	-0.03	-0.25	0.06	0.64
48	8.5	8.48	0.02	1.36	0.04	0.07	0.32	0.26
49	10.5	9.75	0.75	1.27	-0.09	-0.13	-0.20	-0.52
50	10.5	10.93	-0.43	1.18	-0.09	0.00	0.13	0.33
51	11.5	11.92	-0.42	0.99	-0.19	-0.10	-0.10	-0.23
52	12.7	12.62	0.08	0.70	-0.29	-0.10	0.00	0.10
53	13.2	13.27	-0.07	0.65	-0.05	0.24	0.34	0.34
54	13.9	13.50	0.40	0.23	-0.42	-0.37	-0.61	-0.95
55	13.7	13.59	0.11	0.09	-0.14	0.28	0.65	1.26
56	13.2	13.21	-0.01	-0.38	-0.47	-0.33	-0.61	-1.26
57	12.7	12.47	0.23	-0.74	-0.36	0.11	0.44	1.05
58	11.1	11.54	-0.44	-0.93	-0.19	0.17	0.06	-0.38
59	10.4	10.23	0.17	-1.31	-0.38	-0.19	-0.36	-0.42
60	8.9	8.83	0.07	-1.40	-0.09	0.29	0.48	0.84
61	7.5	7.34	0.16	-1.49	-0.09	0.00	-0.29	-0.77
62	6.0	5.76	0.24	-1.58	-0.09	0.00	0.00	0.29
63	3.9	4.55	-0.65	-1.21	0.37	0.46	0.46	0.46
64	3.1	3.30	-0.20	-1.25	-0.04	-0.41	-0.87	-1.33
65	2.0	2.05	-0.05	-1.25	0.00	0.04	0.45	1.32
66	1.1	0.96	0.14	-1.09	0.16	0.16	0.12	-0.33
67	1.3	0.06	1.24	-0.90	0.19	0.03	-0.13	-0.25
68	-1.1	-0.72	-0.38	-0.78	0.12	-0.07	-0.10	0.03
69	-2.0							
70	-1.1							
71	-1.1							
72	-0.4							
73	2.1							

TABLE 13

ADJUSTED VALUES

(STANDARD METHOD - 5th DEGREE - 11 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Difference for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7	9.56	0.14					
7	8.0	8.46	-0.46	-1.10				
8	7.7	7.65	0.05	-0.81	0.29			
9	7.6	7.31	0.29	-0.34	0.47	0.18		
10	7.1	7.20	-0.10	-0.11	0.23	-0.24	-0.42	
11	7.1	7.29	-0.19	0.09	0.20	-0.03	0.21	0.63
12	7.6	7.26	0.34	-0.03	-0.12	-0.32	-0.29	-0.50
13	7.5	7.64	-0.14	0.38	0.41	0.53	0.85	1.14
14	8.3	8.42	-0.12	0.78	0.40	-0.01	-0.54	-1.39
15	9.0	8.92	0.08	0.50	-0.28	-0.68	-0.67	-0.13
16	10.7	10.21	0.49	1.29	0.79	1.07	1.75	2.42
17	11.1	11.23	-0.13	1.02	-0.27	-1.06	-2.13	-3.88
18	11.8	12.25	-0.45	1.02	0.00	0.27	1.33	3.46
19	13.6	13.10	0.50	0.85	-0.17	-0.17	-0.44	-1.77
20	13.5	13.75	-0.25	0.65	-0.20	-0.03	-0.14	0.58
21	14.8	14.43	0.37	0.68	0.03	0.23	0.26	0.12
22	14.4	14.94	-0.54	0.51	-0.17	-0.20	-0.43	-0.69
23	15.3	15.24	0.06	0.30	-0.21	-0.04	0.16	0.59
24	15.6	15.29	0.31	0.05	-0.25	-0.04	0.00	-0.16
25	15.2	14.95	0.27	-0.34	-0.39	-0.14	-0.10	-0.10
26	14.2	14.64	-0.44	-0.31	-0.03	0.42	0.56	0.66
27	13.9	14.21	-0.31	-0.43	-0.12	-0.15	-0.57	-1.13
28	14.1	13.70	0.40	-0.51	-0.08	0.04	0.19	0.76
29	13.2	12.99	0.21	-0.71	-0.20	-0.12	-0.16	-0.35
30	12.0	12.01	-0.01	-0.98	-0.27	-0.07	-0.05	0.21
31	10.2	10.72	-0.52	-1.29	-0.31	-0.04	0.03	-0.02
32	9.6	9.36	0.24	-1.36	-0.07	0.24	0.28	0.25
33	8.2	8.01	0.19	-1.35	0.01	0.08	-0.16	-0.44
34	6.6	6.58	0.02	-1.43	-0.08	-0.09	-0.17	-0.01
35	5.1	5.19	-0.09	-1.39	0.04	0.12	0.21	0.38
36	3.9	4.02	-0.12	-1.17	0.22	0.18	0.06	-0.15
37	3.5	3.51	-0.01	-0.51	0.66	0.44	0.26	0.20
38	3.2	3.25	-0.05	-0.26	0.25	-0.41	-0.85	-1.11
39	3.2	3.02	0.18	-0.23	0.03	-0.22	0.19	1.04
40	2.9	2.70	0.20	-0.32	-0.09	-0.12	0.10	-0.09
41	2.1	2.37	-0.27	-0.33	-0.01	0.08	0.20	0.10
42	2.2	2.31	-0.11	-0.06	0.27	0.28	0.20	0.00
43	2.6	2.45	0.15	0.14	0.20	-0.07	-0.35	-0.55
44	3.3	3.22	0.08	0.77	0.63	0.43	0.50	0.85

TABLE 13
ADJUSTED VALUES

(STANDARD METHCD - 5th DEGREE - 11 VALUE SPREAD)

ITEM	Original Values (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ^{I}	Δ^{II}	Δ^{III}	Δ^{IV}	Δ^{V}
45	4.2	4.25	-0.05	1.03	0.26	-0.37	-0.80	-1.30
46	5.4	5.68	-0.28	1.43	0.40	0.14	0.51	1.31
47	7.6	7.30	0.30	1.62	0.19	-0.21	-0.35	-0.86
48	8.5	8.74	-0.24	1.44	-0.18	-0.37	-0.16	0.19
49	10.5	9.96	0.54	1.22	-0.22	-0.04	-0.33	0.49
50	10.5	10.86	-0.36	0.90	-0.32	-0.10	-0.06	-0.39
51	11.5	11.66	-0.16	0.80	-0.10	-0.22	0.32	0.38
52	12.7	12.57	0.13	0.91	0.11	0.21	-0.01	-0.33
53	13.2	13.20	0.00	0.63	-0.28	-0.39	-0.60	-0.59
54	13.9	13.83	0.07	0.63	0.00	0.28	0.67	1.27
55	13.7	13.73	-0.03	-0.10	-0.73	-0.73	-1.01	-1.68
56	13.2	13.26	-0.06	-0.47	-0.37	0.36	1.09	2.10
57	12.7	12.48	0.22	-0.78	-0.31	0.06	-0.30	-1.39
58	11.1	11.39	-0.29	-1.09	-0.31	0.00	-0.06	0.24
59	10.4	10.31	0.09	-1.08	0.01	0.32	0.32	0.38
60	8.9	8.93	-0.03	-1.38	-0.30	-0.31	-0.63	-0.95
61	7.5	7.39	0.11	-1.54	-0.16	0.14	0.45	1.08
62	6.0	5.89	0.11	-1.50	0.04	0.20	0.06	-0.39
63	3.9	4.13	-0.23	-1.76	-0.26	-0.30	-0.50	-0.56
64	3.1	3.01	0.09	-1.12	0.64	0.90	1.20	1.70
65	2.0	2.20	-0.20	-0.81	0.31	-0.33	-1.23	-2.43
66	1.1	1.28	-0.18	-0.92	-0.11	-0.42	-0.09	1.14
67	1.3	0.33	0.97	-0.95	-0.03	0.08	0.50	0.59
68	-1.1	-0.79	-0.31	-1.12	-0.17	0.14	-0.22	-0.72
69	-2.0							
70	-1.1							
71	-1.1							
72	-0.4							
73	2.1							

TABLE 14
(WEIGHTED METHOD - 3rd DEGREE - 11 VALUE SPREAD

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Difference for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7	9.55	0.15					
7	8.0	8.54	-0.54	-1.01				
8	7.7	7.78	-0.08	-0.76	0.25			
9	7.6	7.34	0.26	-0.44	0.32	0.07		
10	7.1	7.17	-0.07	-0.17	0.27	-0.05	-0.12	
11	7.1	7.17	-0.07	0.00	0.17	-0.10	-0.05	0.07
12	7.6	7.34	0.26	0.17	0.17	0.00	0.10	0.15
13	7.5	7.74	-0.24	0.40	0.23	0.06	0.06	-0.04
14	8.3	8.38	-0.08	0.64	0.24	0.01	-0.05	-0.11
15	9.0	9.23	-0.23	0.85	0.21	-0.03	-0.04	0.01
16	10.7	10.20	0.50	0.97	0.12	-0.09	-0.06	-0.02
17	11.1	11.22	-0.12	1.02	0.05	-0.07	0.02	0.08
18	11.8	12.20	-0.40	0.98	-0.04	-0.09	-0.02	-0.04
19	13.6	13.06	0.54	0.86	-0.12	-0.08	0.01	0.03
20	13.5	13.80	-0.30	0.74	-0.12	0.00	0.08	0.07
21	14.8	14.43	0.37	0.63	-0.11	0.01	0.01	-0.07
22	14.4	14.92	-0.52	0.49	-0.14	-0.03	-0.04	-0.05
23	15.3	15.19	0.11	0.27	-0.22	-0.08	-0.05	-0.01
24	15.6	15.21	0.39	0.02	-0.25	-0.03	0.05	0.10
25	15.2	15.02	0.18	-0.19	-0.21	0.04	0.07	0.02
26	14.2	14.68	-0.48	-0.34	-0.15	0.06	0.02	-0.05
27	13.9	14.25	-0.35	-0.43	-0.09	0.06	0.00	-0.02
28	14.1	13.68	0.42	-0.57	-0.14	-0.05	-0.11	-0.11
29	13.2	12.93	0.27	-0.75	-0.18	-0.04	0.01	0.12
30	12.0	11.94	0.06	-0.99	-0.24	-0.06	-0.02	-0.03
31	10.2	10.73	-0.53	-1.21	-0.22	0.02	0.08	0.10
32	9.6	9.37	0.23	-1.36	-0.15	0.07	0.05	-0.03
33	8.2	7.96	0.24	-1.41	-0.05	0.10	0.03	-0.02
34	6.6	6.55	0.05	-1.41	0.00	0.05	-0.05	-0.08
35	5.1	5.25	-0.15	-1.30	0.11	0.11	0.06	0.11
36	3.9	4.23	-0.33	-1.02	0.28	0.17	0.06	0.00
37	3.5	3.56	-0.06	-0.67	0.35	0.07	-0.10	-0.16
38	3.2	3.18	0.02	-0.38	0.29	-0.06	-0.13	-0.03
39	3.2	2.89	0.31	-0.29	0.09	-0.20	-0.14	-0.01
40	2.9	2.61	0.29	-0.28	0.01	-0.08	0.12	0.26
41	2.1	2.38	-0.28	-0.23	0.05	0.04	0.12	0.00
42	2.2	2.32	-0.12	-0.06	0.17	0.12	0.08	-0.04
43	2.6	2.60	0.00	0.28	0.34	0.17	0.05	-0.03
44	3.3	3.27	0.03	0.67	0.39	0.05	-0.12	-0.17
45	4.2	4.34	-0.14	1.07	0.40	0.01	-0.04	0.08
46	5.4	5.72	-0.32	1.38	0.31	-0.09	-0.10	-0.06

TABLE 14
WEIGHTED METHOD - 3rd DEGREE - 11 VALUE SPREAD

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
47	7.6	7.22	0.38	1.50	0.12	-0.19	-0.10	0.00
48	8.5	8.64	-0.14	1.42	-0.08	-0.20	-0.01	0.09
49	10.5	9.87	0.63	1.23	-0.19	-0.11	0.09	0.10
50	10.5	10.88	-0.38	1.01	-0.22	-0.03	0.08	-0.01
51	11.5	11.78	-0.28	0.90	-0.11	0.11	0.14	0.06
52	12.7	12.58	0.12	0.80	-0.10	0.01	-0.10	-0.24
53	13.2	13.26	-0.06	0.68	-0.12	-0.02	-0.03	0.07
54	13.9	13.66	0.24	0.40	-0.28	-0.16	-0.14	-0.11
55	13.7	13.67	0.03	0.01	-0.39	-0.11	0.05	0.19
56	13.2	13.24	-0.04	-0.43	-0.44	-0.05	0.06	0.01
57	12.7	12.47	0.23	-0.77	-0.34	0.10	0.15	0.09
58	11.1	11.46	-0.36	-1.01	-0.24	0.10	0.00	-0.15
59	10.4	10.26	0.14	-1.20	-0.19	0.05	-0.05	-0.05
60	8.9	8.89	0.01	-1.37	-0.17	0.02	-0.03	0.02
61	7.5	7.38	0.12	-1.51	-0.14	0.03	0.01	0.04
62	6.0	5.81	0.19	-1.57	-0.06	0.08	0.05	0.04
63	3.9	4.34	-0.44	-1.47	0.10	0.16	0.08	0.03
64	3.1	3.12	-0.02	-1.22	0.25	0.15	-0.01	-0.09
65	2.0	2.11	-0.11	-1.01	0.21	-0.04	-0.19	-0.18
66	1.1	1.16	-0.06	-0.95	0.06	-0.15	-0.11	0.08
67	1.3	0.22	1.08	-0.94	0.01	-0.05	0.10	0.21
68	-1.1	-0.68	-0.42	-0.90	0.04	0.03	0.08	-0.02

TABLE 15
(STANDARD METHOD - 3rd DEGREE - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Difference for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7							
7	8.0							
8	7.7							
9	7.6							
10	7.1							
11	7.1							
12	7.6							
13	7.5	8.49	-0.99					
14	8.3	9.00	-0.70	0.51				
15	9.0	9.60	-0.60	0.60	0.09			
16	10.7	10.27	0.43	0.67	0.07	-0.02		
17	11.1	11.04	0.06	0.77	0.10	0.03	0.05	
18	11.8	11.85	-0.05	0.81	0.04	-0.06	-0.09	-0.14
19	13.6	12.66	0.94	0.81	0.00	-0.04	0.02	0.11
20	13.5	13.32	0.18	0.66	-0.15	-0.15	-0.11	-0.13
21	14.8	13.90	0.90	0.58	-0.08	0.07	0.22	0.33
22	14.4	14.37	0.03	0.47	-0.11	-0.03	-0.10	-0.32
23	15.3	14.68	0.62	0.31	-0.16	-0.05	-0.02	0.08
24	15.6	14.80	0.80	0.12	-0.19	-0.03	0.02	0.04
25	15.2	14.70	0.50	-0.10	-0.22	-0.03	0.00	-0.02
26	14.2	14.34	-0.14	-0.36	-0.26	-0.04	-0.01	-0.01
27	13.9	13.77	0.13	-0.57	-0.21	0.05	0.09	0.10
28	14.1	13.04	1.06	-0.73	-0.16	0.05	0.00	-0.09
29	13.2	12.25	0.95	-0.79	-0.06	0.10	0.05	0.05
30	12.0	11.31	0.69	-0.94	-0.15	-0.09	-0.19	-0.24
31	10.2	10.26	-0.06	-1.05	-0.11	0.04	0.13	0.32
32	9.6	9.20	0.40	-1.06	-0.01	0.10	0.06	-0.07
33	8.2	8.07	0.13	-1.13	-0.07	-0.06	-0.16	-0.22
34	6.6	6.99	-0.39	-1.08	0.05	0.12	0.18	0.34
35	5.1	5.86	-0.76	-1.13	-0.05	-0.10	-0.22	-0.40
36	3.9	4.91	-1.01	-0.95	0.18	0.23	0.33	0.55
37	3.5	4.09	-0.59	-0.82	0.13	-0.05	-0.28	-0.61
38	3.2	3.51	-0.31	-0.58	0.24	0.11	0.16	0.44
39	3.2	3.08	0.12	-0.43	0.15	-0.09	-0.20	-0.36
40	2.9	2.84	0.06	-0.24	0.19	0.04	0.13	0.33
41	2.1	2.90	-0.80	0.06	0.30	0.11	0.07	-0.06
42	2.2	3.18	-0.98	0.28	0.22	-0.08	-0.19	-0.26
43	2.6	3.69	-1.09	0.51	0.23	0.01	0.09	0.28
44	3.3	4.37	-1.07	0.68	0.17	-0.06	-0.07	-0.16
45	4.2	5.25	-1.05	0.88	0.20	0.03	0.09	0.16

TABLE 15
(STANDARD METHOD - 3rd. DEGREE - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
46	5.4	6.29	-0.89	1.04	0.16	-0.04	-0.07	-0.16
47	7.6	7.35	0.25	1.06	0.02	-0.14	-0.10	-0.03
48	8.5	8.41	0.09	1.06	0.00	-0.02	0.12	0.22
49	10.5	9.41	1.09	1.00	-0.06	-0.06	-0.04	-0.16
50	10.5	10.35	0.15	0.94	-0.06	0.00	0.06	0.10
51	11.5	11.22	0.28	0.87	-0.07	-0.01	-0.01	-0.07
52	12.7	11.90	0.80	0.68	-0.19	-0.12	-0.11	-0.10
53	13.2	12.38	0.82	0.48	-0.20	-0.01	0.11	0.22
54	13.9	12.57	1.33	0.19	-0.29	-0.09	-0.08	-0.19
55	13.7	12.45	1.25	-0.12	-0.31	-0.02	0.07	0.15
56	13.2	12.19	1.01	-0.26	-0.14	0.17	0.19	0.12
57	12.7	11.67	1.03	-0.52	-0.26	-0.12	-0.29	-0.48
58	11.1	10.84	0.26	-0.83	-0.31	-0.05	+0.07	0.36
59	10.4	9.82	0.58	-1.02	-0.19	0.12	0.17	0.10
60	8.9	8.68	0.22	-1.14	-0.12	0.07	-0.05	-0.22
61	7.5	7.33	0.17	-1.35	-0.21	-0.09	-0.16	-0.11
62	6.0							
63	3.9							
64	3.1							
65	2.0							
66	1.1							
67	1.3							
68	-1.1							
69	-2.0							
70	-1.1							
71	-1.1							
72	-0.4							
73	2.1							

TABLE 16
(STANDARD METHOD - 5th DEGREE - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7							
7	8.0							
8	7.7							
9	7.6							
10	7.1							
11	7.1							
12	7.6							
13	7.5	7.80	-0.30					
14	8.3	8.40	-0.10	0.60				
15	9.0	9.25	-0.25	0.85	0.25			
16	10.7	10.25	0.45	1.00	0.15	-0.10		
17	11.1	11.21	-0.11	0.96	-0.04	-0.19	-0.09	
18	11.8	12.11	-0.31	0.90	-0.06	-0.02	0.17	0.26
19	13.6	12.90	0.70	0.79	-0.11	-0.05	-0.03	-0.20
20	13.5	13.73	-0.23	0.83	0.04	0.15	0.20	0.23
21	14.8	14.40	0.40	0.67	-0.16	-0.20	-0.35	-0.55
22	14.4	14.86	-0.46	0.46	-0.21	-0.05	0.15	0.50
23	15.3	15.15	0.15	0.29	-0.17	0.04	0.09	-0.06
24	15.6	15.26	0.34	0.11	-0.18	-0.01	-0.05	-0.14
25	15.2	15.17	0.03	-0.09	-0.20	-0.02	-0.01	0.04
26	14.2	14.89	-0.69	-0.28	-0.19	0.01	0.03	0.04
27	13.9	14.40	-0.50	-0.49	-0.21	-0.02	-0.03	-0.06
28	14.1	13.69	0.41	-0.71	-0.22	-0.01	0.01	0.04
29	13.2	12.69	0.51	-1.00	-0.29	-0.07	-0.06	-0.07
30	12.0	11.60	0.40	-1.09	-0.09	0.20	0.27	0.33
31	10.2	10.47	-0.27	-1.13	-0.04	0.05	-0.15	-0.42
32	9.6	9.22	0.38	-1.25	-0.12	-0.08	-0.13	0.02
33	8.2	8.01	0.19	-1.21	0.04	0.16	0.24	0.37
34	6.6	6.78	-0.18	-1.23	-0.02	-0.06	-0.22	-0.46
35	5.1	5.71	-0.61	-1.07	0.16	0.18	0.24	0.46
36	3.9	4.65	-0.75	-1.06	0.01	-0.15	-0.33	-0.57
37	3.5	3.71	-0.21	-0.94	0.12	0.11	0.26	0.59
38	3.2	2.87	0.33	-0.84	0.10	-0.02	-0.13	-0.39
39	3.2	2.36	0.84	-0.51	0.33	0.23	0.25	0.38
40	2.9	2.18	0.72	-0.18	0.33	0.00	-0.23	-0.48
41	2.1	2.21	-0.11	0.03	0.21	0.12	-0.12	0.11
42	2.2	2.50	-0.30	0.29	0.26	0.05	0.17	0.29
43	2.6	3.00	-0.40	0.50	0.21	-0.05	-0.10	-0.27
44	3.3	3.78	-0.48	0.78	0.28	0.07	0.12	0.22

TABLE 16
(STANDARD METHOD - 5th DEGREE - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
45	4.2	4.71	-0.51	0.93	0.15	-0.13	-0.20	-0.32
46	5.4	5.76	-0.36	1.05	0.12	-0.03	0.10	0.30
47	7.6	6.97	0.63	1.21	0.16	0.04	0.07	-0.03
48	8.5	8.26	0.24	1.29	0.08	-0.08	-0.12	-0.19
49	10.5	9.60	0.90	1.34	0.05	-0.03	0.05	0.17
50	10.5	10.87	-0.37	1.27	-0.07	-0.12	-0.09	-0.14
51	11.5	11.94	-0.44	1.07	-0.20	-0.13	-0.01	0.08
52	12.7	12.81	-0.11	0.87	-0.20	0.00	0.13	0.14
53	13.2	13.35	-0.15	0.54	-0.33	-0.13	-0.13	-0.26
54	13.9	13.58	0.32	0.23	-0.31	0.02	0.15	0.28
55	13.7	13.52	0.18	-0.06	-0.29	0.02	0.00	-0.15
56	13.2	13.00	0.20	-0.52	-0.46	-0.17	-0.19	-0.19
57	12.7	12.24	0.46	-0.76	-0.24	0.22	0.39	0.58
58	11.1	11.36	-0.26	-0.88	-0.12	0.12	-0.10	-0.49
59	10.4	10.28	0.12	-1.08	-0.20	-0.08	-0.20	-0.10
60	8.9	8.99	-0.09	-1.29	-0.21	-0.01	0.07	0.27
61	7.5	7.66	-0.16	-1.33	-0.04	0.17	0.18	0.11
62	6.0							
63	3.9							
64	3.1							
65	2.0							
66	1.1							
67	1.3							
68	-1.1							
69	-2.0							
70	-1.1							
71	-1.1							
72	-0.4							
73	2.1							

TABLE 17
(WEIGHTED METHOD - 3rd DEGREE - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
1	14.1							
2	13.0							
3	12.6							
4	12.0							
5	10.8							
6	9.7							
7	8.0							
8	7.7							
9	7.6							
10	7.1							
11	7.1							
12	7.6							
13	7.5	8.00	-0.50					
14	8.3	8.61	-0.31	0.61				
15	9.0	9.37	-0.37	0.76	0.15			
16	10.7	10.23	0.47	0.86	0.10	-0.05		
17	11.1	11.14	-0.04	0.91	0.05	-0.05	0.00	
18	11.8	12.04	-0.24	0.90	-0.01	-0.06	-0.01	-0.01
19	13.6	12.89	0.71	0.85	-0.05	-0.04	0.02	0.03
20	13.5	13.64	-0.14	0.75	-0.10	-0.05	-0.01	-0.03
21	14.8	14.26	0.54	0.62	-0.13	-0.03	0.02	0.03
22	14.4	14.72	-0.32	0.46	-0.16	-0.03	0.00	-0.02
23	15.3	15.00	0.30	0.28	-0.18	-0.02	0.01	0.01
24	15.6	15.10	0.50	0.10	-0.18	0.00	0.02	0.01
25	15.2	15.00	0.20	-0.10	-0.20	-0.20	-0.20	-0.04
26	14.2	14.70	-0.50	-0.30	-0.20	0.00	0.02	0.04
27	13.9	14.20	-0.30	-0.50	-0.20	0.00	0.00	-0.02
28	14.1	13.50	0.60	-0.70	-0.20	0.00	0.00	0.00
29	13.2	12.62	0.58	-0.88	-0.18	0.02	0.02	0.02
30	12.0	11.59	0.41	-1.03	-0.15	0.03	0.01	-0.01
31	10.2	10.45	-0.25	-1.14	-0.11	0.04	0.01	0.00
32	9.6	9.23	0.37	-1.22	-0.08	0.03	-0.01	-0.02
33	8.2	8.00	0.20	-1.23	-0.01	0.07	0.04	0.05
34	6.6	6.80	-0.20	-1.20	0.03	0.04	-0.03	-0.07
35	5.1	5.67	-0.57	-1.13	0.07	0.04	0.00	0.03
36	3.9	4.66	-0.76	-1.01	0.12	0.05	0.01	0.01
37	3.5	3.81	-0.31	-0.85	0.16	0.04	-0.01	-0.02
38	3.2	3.13	0.07	-0.68	0.17	0.01	-0.03	-0.02
39	3.2	2.66	0.54	-0.47	0.21	0.04	0.03	0.06
40	2.9	2.43	0.47	-0.23	0.24	0.03	-0.01	-0.04
41	2.1	2.43	-0.33	0.00	0.23	-0.01	-0.04	-0.03
42	2.2	2.67	-0.47	0.24	0.24	0.01	0.02	0.06
43	2.6	3.16	-0.56	0.49	0.25	0.01	0.00	-0.02
44	3.3	3.88	-0.58	0.72	0.23	-0.02	-0.03	-0.03
45	4.2	4.80	-0.60	0.92	0.20	-0.03	-0.01	0.02
46	5.4	5.89	-0.49	1.09	0.17	-0.03	0.00	0.01
47	7.6	7.09	0.51	1.20	0.11	-0.06	-0.03	-0.03

TABLE 17
(WEIGHTED METHOD - 3rd Degree - 25 VALUE SPREAD)

ITEM	Original Value (Degrees)	Adjusted Value (Degrees)	Residual (Degrees)	Differences for Adjusted Values				
				Δ'	Δ''	Δ'''	Δ^{IV}	Δ^{V}
48	8.5	8.35	0.15	1.26	0.06	-0.05	0.01	0.04
49	10.5	9.59	0.91	1.24	-0.02	-0.08	-0.03	-0.04
50	10.5	10.74	-0.24	1.15	-0.09	-0.07	0.01	0.04
51	11.5	11.74	-0.24	1.00	-0.15	-0.06	0.01	0.00
52	12.7	12.52	0.18	0.78	-0.22	-0.07	-0.01	-0.02
53	13.2	13.04	0.16	0.52	-0.26	-0.04	0.03	0.04
54	13.9	13.27	0.63	0.23	-0.29	-0.03	0.01	-0.02
55	13.7	13.20	0.50	-0.07	-0.30	-0.01	0.02	0.01
56	13.2	12.82	0.38	-0.38	-0.31	-0.01	0.00	-0.02
57	12.7	12.16	0.54	-0.66	-0.28	0.03	0.04	0.04
58	11.1	11.25	-0.15	-0.91	-0.25	0.03	0.00	-0.04
59	10.4	10.13	0.27	-1.12	-0.21	0.04	0.01	0.01
60	8.9	8.85	0.05	-1.28	-0.16	0.05	0.01	0.00
61	7.5	7.46	0.04	-1.39	-0.11	0.05	0.00	-0.01
62	6.0							
63	3.9							
64	3.1							
65	2.0							
66	1.1							
67	1.3							
68	-1.1							
69	-2.0							
70	-1.1							
71	-1.1							
72	-0.4							
73	2.1							

TABLE 18
ROOT MEAN SQUARE VALUES

Spread		11 Point Smoothing			25 Point Smoothing		
Type	Original	Standard		Weighted	Standard		Weighted
Order		3rd. Order	5th Order	3rd. Order	3rd. Order	5th. Order	3rd. Order
Residual		0.3578	0.2637	0.2936	0.7122	0.4196	0.4356
Δ'	1.0370	0.9045	0.9272	0.9104	0.7741	0.8848	0.8552
Δ''	0.8750	0.2292	0.2936	0.2175	0.1719	0.1995	0.1815
Δ'''	1.5711	0.1878	0.3361	0.0928	0.0850	0.1103	0.0404
Δ^{IV}	2.9811	0.3161	0.5652	0.0781	0.1382	0.1721	0.0191
Δ^{V}	5.7071	0.5802	0.9743	0.0951	0.2535	0.3123	0.0307

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