AD No. 13 338
ASTIA FILE COPY

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VI, 103–156 (1953)

On Nonlinear Elliptic Partial Differential Equations and Hölder Continuity

By LOUIS NIRENBERG

1. *Introduction*

1. This paper is concerned with general nonlinear elliptic partial differential equations of second order for functions of two independent variables. New a priori estimates for the derivatives of solutions of such equations are derived and used to obtain various results. In particular, a proof is given for the existence of solutions of the boundary value problem for quasilinear elliptic equations in convex domains.

The basic device used in deriving the a priori estimates is a lemma (Lemma 2 of Section 3) expressing a relation between the distortion of a class of mappings of a domain in the plane—this class includes the quasi-conformal mappings—and the Hölder continuity[1] of the mappings in that class. The techniques used here are closely related to those of C. B. Morrey in [13] and in his work on multiple integral variational problems in the Calculus of Variations [14]. In addition, extensive use is made of the theory of linear elliptic partial differential equations as developed by J. Schauder in [19].

Some of the theorems proved here have been extended to elliptic equations in more than two independent variables; this extension will be presented in a forthcoming paper.

We proceed to describe our main results.

2. One of the aims of this paper is to derive a priori estimates for the derivatives of solutions $z(x,y)$ of a general nonliear elliptic equation

$$F(x,y,z,p,q,r,s,t) = 0, \qquad 4F_rF_t - F_s^2 > 0. \tag{1.1}$$

Here $p, \cdots, t$ represent the partial derivatives of z, $p = z_x, \cdots, t = z_{yy}$, and $F_r = \partial F/\partial r$ etc. The principal theorem to be proved in this connection, from which the others follow by use of known theorems for *linear* elliptic equations, is

[1]A function f defined in a set is said to satisfy a Hölder condition, or inequality, in that set if there exist two positive constants C, α ($\alpha < 1$) such that for any two points P, P' of the set the inequality $|f(P) - f(P')| \leq C\overline{PP'}^{\alpha}$ holds, where $\overline{PP'}$ denotes the distance between P and P'. The constants C and α are called the coefficient and exponent, or simply the constants, of the Hölder condition. A function satisfying a Hölder condition is sometimes said to be Hölder continuous, its Hölder continuity being described by the Hölder inequality.

Theorem I: Let $z(x,y)$ be a function defined in a domain[2] $\mathfrak{D}$ in the x,y-plane satisfying there the elliptic partial differential equation (1.1).

Assume: (i) F has continuous first derivatives with respect to its eight arguments (x,y,z,p,q,r,s,t) in an open set of the eight dimensional space containing the hypersurface $(x,y,z(x,y), \cdots ,t(x,y))$ given by the solution $z(x,y)$. The first partial derivatives of F on this hypersurface are bounded in absolute value by a constant K.

(ii) $z(x,y)$ has continuous first and second derivatives in $\mathfrak{D}$ which are bounded in absolute value by a constant K_1.

(iii) For any real numbers ξ,η the inequality

$$F_r\xi^2 + F_s\xi\eta + F_t\eta^2 \geq \lambda(\xi^2 + \eta^2) \tag{1.2}$$

holds for all x,y in $\mathfrak{D}$, where λ is a positive constant.

Conclusion: In any closed subdomain $\mathfrak{B}$, of $\mathfrak{D}$, the second derivatives of $z(x,y)$ satisfy a Hölder condition whose coefficient and exponent depend only on K, K_1, λ, and the distance from the closed subdomain $\mathfrak{B}$ to the boundary of $\mathfrak{D}$[3].

Additional a priori estimates for all derivatives of a solution of (1.1) in terms of bounds of its derivatives of first and second order are derived, using Theorem I, in §9. These estimates include those obtained by S. Bernstein [2] and J. Schauder ([18], Sections 4,6) for solutions of equations of the form (1.1). Our derivation of the estimates from Theorem I uses results concerning just *linear* elliptic equations—whereas their derivation involves the 'auxiliary function' of Bernstein. (Further references and remarks concerning the derivation of a priori estimates for derivatives of solutions of (1.1) are given in §9, 1.)

Theorem I was developed principally in order to establish existence theorems for nonlinear elliptic equations; it has been applied to solve the Weyl and Minkowski problems in differential geometry in the large. (The solutions of these problems will appear in a forthcoming issue.) The strength of the theorem lies in the nature of the Hölder condition arising in its conclusion. As an illustration of the manner in which it is used we deduce the following theorem concerning compactness of solutions of nonlinear elliptic equations.

Theorem II. Let $z_n(x,y)$, $n = 1, 2, \cdots$, be a sequence of functions defined in a bounded domain $\mathfrak{D}$ in the x,y-plane, satisfying elliptic partial differential

---

[2]The term domain is always used to denote an open point set. The term closed domain denotes the closure of a domain. A point set is said to be a closed subdomain of a domain $\mathfrak{A}$ if it is a closed domain and is contained in $\mathfrak{A}$.

[3]This result has also been extended to the general second order elliptic equation of the type (1.1) with any number of independent variables; but the final result for this general case is not as strong as Theorem I, for, the coefficient of the derived Hölder inequality depends—in addition to the bounds for the second derivatives of the solution—on the modulus of continuity of the second derivatives.

equations

$$F_n(x, y, z_n, p_n, q_n, r_n, s_n, t_n) = 0$$

in $\mathfrak{D}$.

Assume: There exist positive constants K, K_1, λ, independent of n, such that z_n and F_n satisfy the conditions (i), (ii) and (iii) of Theorem I with respect to these constants. Assume further that the z_n are uniformly bounded in absolute value in $\mathfrak{D}$.

Conclusion: There exists a subsequence of the z_n which converges in $\mathfrak{D}$ to a function $z(x,y)$ having continuous first and second derivatives; these are the limits of the corresponding derivatives of the members of the subsequence. Furthermore, if the functions F_n converge to a function F then the limit function z is a solution of the limit differential equation

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0.$$

Proof: From Theorem I it follows that the second derivatives of the z_n satisfy a uniform Hölder condition (independent of n) in any closed subdomain of $\mathfrak{D}$. They are therefore equicontinuous in this subdomain. Because of the uniform boundedness of the z_n and their first and second derivatives it follows that we may select a subsequence z_{n_i}, of the z_n, such that the z_{n_i} and their first and second derivatives converge in this subdomain. Since this is true for every closed subdomain of $\mathfrak{D}$ we can—by choosing a suitable sequence of closed subdomains and by the usual diagonalization process—find the subsequence of the z_n which will converge (together with first and second derivatives) to a function z (and its corresponding derivatives) in all of $\mathfrak{D}$. The convergence is uniform in any closed subdomain. The last statement of Theorem II follows immediately.

3. Theorem I is more than a theorem on a priori bounds of solutions of elliptic equations. The Hölder continuity of the second derivatives of the solution $z(x,y)$ of (1.1) is not assumed: it is derived as a consequence of the other assumptions. For this reason the theorem is also of interest in connection with the question of analyticity of solutions of elliptic partial differential equations. With its aid one may weaken the conditions under which the analyticity of a solution of (1.1) may be inferred when F is analytic in its arguments.

The fundamental question concerning the differentiability and analyticity of solutions of analytic elliptic partial differential equations has received considerable attention since the classical work of S. Bernstein [1]—the general elliptic system of equations, with any number of both dependent and independent variables was finally treated by I. G. Petrovsky [15]. He proved the analyticity of sufficiently often differentiable solutions of such analytic systems.

However, in all the proofs of the analyticity of solutions it is not sufficient to assume that the solutions have continuous derivatives up to the orders that occur in the equations. Usually further differentiability conditions are required.

Bernstein's proof [1] of the analyticity of a solution $z(x,y)$ of the general second order elliptic equation (1.1), where F is analytic in all its arguments, makes use of the assumption that z has continuous third derivatives. Later this result was obtained by M. Gevrey [3], (see [12], pages 1320–1324, for further references), and again by H. Lewy [10] who extended the solution z and the equation (1.1) to complex values of the arguments x,y. Other proofs of the analyticity of any solution z of a nonlinear elliptic equation of second order with *any* number of independent variables were given by G. Giraud [4] and E. Hopf [5], under the weaker assumption that z has continuous first and second derivatives and that the second derivatives satisfy Hölder conditions. They also proved ([5], Theorems IV and V, pages 211–215), under the same assumption on z, that if the function F has partial deriviatives with respect to all of its arguments up to order m which satisfy Hölder conditions, then the solution z of (1.1) possesses derivatives up to order $m+2$ satisfying Hölder conditions. The question—whether the continuity alone of the first and second derivatives of z implies that z is analytic (when F is)—has remained open.

Theorem I settles this question and together with the results of Giraud and Hopf implies

Theorem III:[4] Let $z(x,y)$ have continuous first and second derivatives and satisfy an elliptic partial differential equation

$$F(x, y, z, p, q, e, s, t) = 0$$

in a domain in the x,y-plane. Then

(a) if F is an analytic function of its eight arguments then $z(x,y)$ is an analytic function;

(b) if F has continuous partial derivatives with respect to its eight arguments up to order m which satisfy Hölder conditions (in these arguments), then z possesses continuous partial derivatives up to order $m + 2$ satisfying Hölder conditions.

4. The proof of Theorem I is based on a result of C. B. Morrey concerning linear elliptic equations (see Lemma 1 of Section 6, [13]). We give a new and more direct proof of this important result by a method which has the further advantage of admitting a generalization to more independent variables. The result is

Theorem IV: Let $z(x,y)$ be defined in a domain $\mathcal{G}$ in the x,y-plane and satisfy the elliptic partial differential equation

$$Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0. \tag{1.3}$$

Assume: (i) The coefficients A, B, C and D are functions of (x,y) bounded in absolute value by a constant K.

[4]Because of the result mentioned in footnote 3 this theorem may be extended to solutions of second order nonlinear elliptic equations in any number of independent variables.

(ii) $z(x,y)$ has continuous first and second derivatives in $\mathfrak{A}$ and the first derivatives are bounded in absolute value by a constant K_1.

(iii) For any real numbers ξ,η the inequality

$$A\xi^2 + B\xi\eta + C\eta^2 \geq \lambda(\xi^2 + \eta^2) \tag{1.4}$$

holds for all (x,y) in $\mathfrak{A}$; here λ is a positive constant.

Conclusion: In any closed subdomain $\mathfrak{B}$, of $\mathfrak{A}$, the first derivatives of $z(x,y)$ satisfy a Hölder inequality with constants depending only on K, K_1, λ, and the distance from the closed subdomain $\mathfrak{B}$ to the boundary of $\mathfrak{A}$[5].

Theorem IV yields an estimate for the Hölder continuity of the *first* derivatives of z, using, essentially, only the boundedness of the coefficients A, B, C, D. E. Hopf ([15], Theorem I, page 208) has shown that if the coefficients considered as functions of (x,y) satisfy a Hölder condition, then Hölder inequalities for the *second* derivatives of z in any closed subdomain may also be derived.

Since so little is assumed about the coefficients A, B, C and D—just that they are bounded and satisfy (1.4)—Theorem IV is useful in studying nonlinear elliptic equations. For example, the coefficients A, B, C, D may already involve $z(x,y)$ and derivatives of $z(x,y)$ of any order, so that (1.3) may be nonlinear. In fact, Theorem IV is employed in §8 to derive an existence theorem for quasilinear elliptic equations. There we prove the existence of a solution $z(x,y)$ of the boundary value problem for the general quasilinear elliptic equation of the form

$$A(x, y, z, z_x, z_y)z_{xx} + B(x, y, z, z_x, z_y)z_{xy} + C(x, y, z, z_x, z_y)z_{yy} = 0 \tag{1.5}$$

in a convex domain in the plane.

Such an existence theorem was obtained by J. Leray and J. Schauder [9] as an application of their concept of degree of mapping in Banach space.

In order to solve the boundary value problem for equation (1.5) one must derive a priori estimates for its solutions. The interesting feature of our proof of the existence of a solution is that, in obtaining a priori estimates, we use only results concerning *linear* elliptic equations. This avoids the involved procedure (due to Bernstein) used by Schauder to obtain a priori estimates for second derivatives of a solution. (See [18], Section 4, where reference is made to pages 119–125 of [2].) In addition, the nature of the a priori bounds obtained here is such that the notion of degree of a mapping in Banach space is not needed. Instead we use a fixed point theorem concerning transformations in Banach space due to Schauder [17]. Finally, we remark that the existence theorem in [9] assumes more of the differential equation than our theorem in §8.

Morrey [13] observed that Theorem IV could be used to show the existence of a solution of (1.5) but his proof contains a gap. Further remarks about the work of Leray and Schauder, and Morrey, are made in the Outline of §8, 1.

Our existence proof makes use of Schauder's work [19] on linear second

[5]See the end of §4 for more general conditions under which the conclusion still holds.

order elliptic equations. In order to be able to use his existence theorems concerning such equations we derive in §6 a sharp form of Theorem IV which is expressed as Theorem V, and which is of some interest in itself (see remarks in §6, 1). Theorem V contains extra conditions imposed on the solution $z(x,y)$ of (1.3) enabling one to calculate a Hölder inequality for the first derivatives of z in the whole domain $\mathfrak{A}$. These extra conditions are of the nature of assumptions about the boundary of $\mathfrak{A}$ and the boundary values of $z(x,y)$.

5. The proofs of the results described above are presented in §4, 5, 8 and 9—the remaining sections contain subsidiary lemmas which are employed in the proofs. At the end of some sections remarks are inserted which show how the principal results of the section may be strengthened. These remarks are not used in our discussion of partial differential equations.

Theorem IV is derived as a simple consequence of a fundamental lemma, Lemma 2 of §3, which concerns the Hölder continuity of a class of mappings (including quasi-conformal mappings). This lemma and a few related lemmas concerning these mappings are proved in §3 with the aid of another lemma (suggested by K. O. Friedrichs), Lemma 1, of §2, 1. The proof of Theorem IV, using Lemma 2, is then given in §4. The techniques used in these sections, §2–4, are modifications of those developed by Morrey in his work on multiple integral variational problems [14], and are, together with the proof of Theorem I related to those employed by M. Shiffman in his proof of the analyticity of solutions of multiple integral variational problems [20]. The proof of Theorem I is given in §5 and consists in transforming equation (1.1) into equations similar to (1.3) for the difference quotients of the solution of (1.1). Thus, for the proofs of Theorem IV (and I) it is sufficient to read §2, 1, §3, §4 (and §5), which are independent of the rest of the paper.

In §6 we prove Theorem V, the sharp form of Theorem IV, using (i) a modification, Lemma 3′, of one of the lemmas on quasi-conformal mappings, which in turn is proved in §7, and (ii) a sharp form of Lemma 1, Lemma 1′, which is proved in §2, 3.

Section 8 treats the quasilinear elliptic equation (1.5). Using Theorem V, and Schauder's theory of linear elliptic equations [19], we prove there the existence of a solution of the boundary value problem in Theorem VI and derive a priori estimates for all solutions. At the end of §8, in No. 8, we show how the existence of a solution of the problem may be derived using Theorem IV instead of Theorem V. This requires a slight modification of an existence theorem for linear elliptic equations, due to Schauder. This whole section is completely independent of the rest of the paper except for reference to Theorems IV and V.

Finally in §9 we derive a priori estimates for derivatives of order greater than two of a solution of (1.1) in a domain $\mathfrak{D}$ in terms of bounds for the derivatives of first and second order. The estimates for the derivatives (of higher order) in closed subdomains of $\mathfrak{D}$ follow from Theorem I with the aid of the theory of linear elliptic equations in [19]. In order to obtain such estimates in the whole

domain $\mathfrak{D}$ we derive first a sharp form of Theorem I, Theorem VII. This consists in imposing conditions on the boundary of $\mathfrak{D}$ and on the boundary values of the solution z, which make it possible to conclude that the second derivatives of z satisfy a Hölder inequality in all of $\mathfrak{D}$.

2. *A Lemma*

1. *Domains.* A large part of this paper will deal with the derivation of estimates for solutions of elliptic differential equations defined in domains $\mathfrak{D}$ in the x,y-plane. The estimates to be obtained are of two kinds: estimates of values of solutions in closed subdomains of $\mathfrak{D}$ and estimates of values in the whole domain $\mathfrak{D}$. The domains of definition $\mathfrak{D}$ of functions for which estimates of the first kind will be derived may be arbitrary open sets in the plane, which we sometimes assume to be bounded. Estimates of the second kind, however, will be derived only for functions in domains satisfying the following condition: $\mathfrak{D}$ is bounded by a finite number of simple closed curves which do not intersect; each curve has a finite length and may be represented by functions $x(s)$, $y(s)$ of arc length s, having continuous derivatives up to order m (an integer).

Definition: Domains having this property are said to be of type L_m.

It is clear that the boundary curves of a [illegible] of type L_2 have bounded curvature. In addition, in a neighborhood of [illegible] on such a curve one of the coordinates, say x, may be introduced as [illegible]cal parameter, so that the curve may be represented (locally) by the equation $y = f(x)$, where $f(x)$ is twice continuously differentiable.

2. Many of the estimates that will concern us are of the nature of Hölder inequalities (see footnote 1, page 103) for functions, as in Theorem IV; to derive such estimates for a function we need a means of estimating the difference of the values of the function at any two points in terms of the distance between the points. The technique we will employ is to establish estimates for certain double integrals involving the derivatives of the function, and then to derive from these estimates the required Hölder inequality for the function.

Of course the estimates of the double integrals must be of such a kind as to imply a Hölder inequality for the function. It is well known that having a *bound* for the Dirichlet integral of a function does *not* enable one to estimate the difference of the values of the function at two points; something stronger is needed. There are some integrals, involving the first derivatives of a function which, together with appropriate estimates, have the required nature. In terms of these estimates the calculation of the constants of the Hölder inequalities is in general not difficult. The difficulty which occurs in practice, in trying to employ one of these integrals in order to derive a Hölder inequality for a function, arises in the attempt to establish the appropriate estimates for the integral.

In his work on multiple integral variational problems C. B. Morrey derived Hölder inequalities for functions by establishing estimates of the "growth" of the Dirichlet integrals of the functions over circles—as a function of the radius.

The fact that such growth estimates imply Hölder inequalities was shown by Morrey in a lemma stating (essentially) that if the Dirichlet integral of a function over every circle is bounded by Kr^α, where K and $\alpha < 1$ are positive constants and r is the radius of the circle (this describes the growth), then the function satisfies a Hölder inequality with constants depending on K and α (see [14], Theorem (2.1) of Chapter 2).

The original proof of Theorem IV of §1 devised by the author was more closely related to Morrey's proof than that to be presented here. There, as with Morrey, the Hölder inequalities for the derivatives p,q of the solution z of (1.3) were obtained by the derivation of estimates of the "growths" of the Dirichlet integrals of p and q (in a manner more direct than Morrey's) and by use of Morrey's lemma. Later K. O. Friedrichs observed that appropriate estimates for another integral, which would imply Hölder continuity for p and q, could be obtained somewhat more simply; we shall follow his procedure here.

The integral to be used is expressed in (2.1), and we prove the analogue of Morrey's lemma, i.e. that estimates for the integral yield Hölder continuity, in the following lemma. It is expressed in a form suited for application in §3 where we shall derive Hölder inequalities for (classes of) functions in closed subdomains of the domain of definition. In No. 3 of this section the lemma is stated in a sharp form suited for application in §6 where Hölder inequalities for functions in the whole domain of definition are derived.

Lemma 1: Let $p(x,y)$ be a function having continuous first derivatives defined in a domain $\mathfrak{A}$ in the x,y-plane. Let $\mathfrak{B}$ be a closed subdomain of $\mathfrak{A}$ and denote its distance from the boundary of $\mathfrak{A}$ by $2d$. Assume that $p(x,y)$ is bounded in absolute value by a positive constant K_1 in $\mathfrak{B}$, and that there exist positive constants M, α, $\alpha < 1$, such that for any circle C_d with center in $\mathfrak{B}$ and radius d the following inequality holds

$$\iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2)\, dA \leq M. \tag{2.1}$$

Here dA represents the element of area in the circle C_d and r the distance of the point of integration from the center of the circle. Then the function $p(x,y)$ satisfies a Hölder condition in $\mathfrak{B}$ with exponent $\alpha/2$ and coefficient depending only on K_1, M, α and d.

The proof of the lemma is not difficult but we present all details so that they may be referred to in the proof of the sharp form in No. 3.

Proof: We wish to show that there exists a constant H depending only on K_1, M, α and d such that for any two points P and P' of $\mathfrak{B}$ the inequality

$$\frac{|p(P) - p(P')|}{\overline{PP'}^{\alpha/2}} \leq H$$

holds, where $\overline{PP'}$ is the distance between P and P'. Clearly if the distance

$\overline{PP'}$ between two points P and P' of $\mathcal{B}$ is not less than d then, since $|p(x,y)| \leq K_1$,

$$\frac{|p(P) - p(P')|}{\overline{PP'}^{\alpha/2}} \leq \frac{2K_1}{d^{\alpha/2}}.$$

We need, therefore, consider only the case where the distance $\overline{PP'} = s$ between two points P and P' of $\mathcal{B}$ is less than d.

For any point (x,y) in $\mathcal{A}$

$$|p(P) - p(P')| \leq |p(P) - p(x,y)| + |p(x,y) - p(P')|.$$

Letting the point (x,y) range over the circle with the line joining P to P' as *diameter*, we integrate both sides of the inequality with respect to (x,y) over this circle (of diameter s) to obtain the inequality

$$\begin{aligned} &\frac{\pi s^2}{4} |p(P) - p(P')| \\ (2.2) \qquad &\leq \iint |p(P) - p(x,y)|\, dx\, dy + \iint |p(x,y) - p(P')|\, dx\, dy. \end{aligned}$$

The first term on the right is certainly not decreased if we enlarge the domain of integration to a circle C_s with center P and radius s. Since $s < d$, C_s lies in $\mathcal{A}$. Introducing polar coordinates (r,θ) in this circle, and denoting by $p(r,\theta)$ the value of p at (r,θ), we note— since $p(r,\theta) - p(P) = \int_0^r p_\rho(\rho,\theta)\, d\rho$ —that the first term is not greater than

$$I_1 = \iint_{C_s} \left[\int_0^r |p_\rho(\rho,\theta)|\, d\rho\right] r\, dr\, d\theta.$$

We now proceed to obtain appropriate estimates for I_1. Integrating by parts with respect to r, the integral I_1 may be written in the form

$$I_1 = \frac{1}{2}\int_0^s \int_0^{2\pi} (s^2 - r^2)\, |p_r|\, dr\, d\theta.$$

Hence

$$I_1 \leq \frac{1}{2} s^2 \int_0^s \int_0^{2\pi} |p_r|\, dr\, d\theta = \frac{1}{2} s^2 \iint_{C_s} \frac{1}{r} |p_r|\, dA$$

where $dA = r\, dr\, d\theta$,

$$\begin{aligned} &= \frac{1}{2} s^2 \iint_{C_s} r^{\alpha/2-1} |p_r|\, r^{-\alpha/2}\, dA \\ &\leq \frac{1}{2} s^2 \left[\iint_{C_s} r^{\alpha-2}\, dA\right]^{1/2} \left[\iint_{C_s} r^{-\alpha} p_r^2\, dA\right]^{1/2}, \end{aligned}$$

by Schwarz's inequality,

$$\leq \frac{1}{2} s^2 \left[\frac{2\pi}{\alpha} s^{\alpha} \right]^{1/2} \left[\iint_{C_s} r^{-\alpha}(p_x^2 + p_y^2)\, dA \right]^{1/2};$$

finally, by (2.1), since the last integral is not decreased if the integration is extended over a circle concentric with C_s and radius d

$$I_1 \leq \sqrt{\frac{\pi M}{2\alpha}}\, s^{2+\alpha/2}. \tag{2.3}$$

The same estimate may be obtained for the second term on the right side of (2.2). Therefore (2.2) yields the inequality

$$\frac{\pi s^2}{4} \mid p(P) - p(P') \mid \leq \sqrt{\frac{2\pi M}{\alpha}}\, s^{2+\alpha/2}$$

or

$$\frac{\mid p(P) - p(P') \mid}{\overline{PP'}^{\alpha/2}} \leq 4\sqrt{\frac{2M}{\pi\alpha}}$$

since $\overline{PP'} = s$. The right side is clearly a constant depending only on M and α, and the proof of the lemma is complete.

Remark: Clearly it is not necessary to assume that the first derivatives of p are continuous; one need merely assume that the integrals occurring in (2.1) exist and satisfy (2.1). This follows from the fact that we may approximate $p(x,y)$ by functions p_n with continuous first derivatives, establish the Hölder condition for these functions, as above, and then let $n \to \infty$ to obtain the required Hölder condition for $p(x,y)$—after some slight argument.

In addition, if the domain $\mathfrak{A}$ and the set $\mathfrak{B}$ are bounded connected sets, it may be shown that the coefficient in the Hölder condition of $p(x,y)$ depends only on M, α, d, and the diameter of $\mathfrak{A}$, and is thus independent of K_1.

3. The following is a sharp form of Lemma 1 to be used in §6:

Lemma 1′: Let $p(x,y)$ be a function having continuous first derivatives defined in a domain $\mathfrak{A}$ of type L_2 (see No. 1). Assume that $\mid p(x,y) \mid \leq K_1$ and assume that there exist positive constants d, M, α, $\alpha < 1$, such that for any circle with center in $\mathfrak{A}$ and radius d the following inequality holds:

$$\iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2)\, dA \leq M. \tag{2.4}$$

Here the integration is extended over the intersection C_d of $\mathfrak{A}$ with the circle; dA represents element of area and r the distance of the point of integration from the center of the circle.

Then the function $p(x,y)$ satisfies a Hölder condition in $\mathfrak{A}$ with exponent $\alpha/2$ and a coefficient that depends only on K_1, M, α, d, and the domain $\mathfrak{A}$.

Proof: The proof is similar to that of Lemma 1 in No. 2, and in order to make use of the calculations performed there, we adopt the following, seemingly artificial, procedure.

Let P be any point of $\mathfrak{A}$. Consider a domain in the shape of a quarter circle with P at the vertex, that is, a domain bounded by a circular arc with P as center, and by two mutually perpendicular radii from P. If the length of the radius of the circle is s we call such a domain $\Omega(P,s)$. From the fact that the boundary curves of $\mathfrak{A}$ have no corners it follows that for any point P of $\mathfrak{A}$ one can find such a domain $\Omega(P,s)$ which lies entirely in $\mathfrak{A}$, at least for s sufficiently small[6]. Clearly for any point P and radius s there may exist many such domains.

From the fact that $\mathfrak{A}$ is of type L_2 (in particular, since the curvatures of the boundary curves are bounded) it follows that there exists a positive number d', which we can assume to be less than d, such that for any two points P and P' of $\mathfrak{A}$ whose distance from each other, which we denote by s, is less than d', there exist two such domains $\Omega(P,s)$, $\Omega(P',s)$ lying entirely in $\mathfrak{A}$ with the property that the area of their intersection (which may be greater than $s^2/2$) is not less than $s^2/4$.

We use domains $\Omega(P,s)$ because the calculations performed on pages 111 and 112 for integrals in circles C_s may be carried over for integrals in these domains.

We now proceed to establish the Hölder condition for p. We must show that there exists a constant H depending only on K_1, M, α, d, and the domain $\mathfrak{A}$, such that for any two points P and P' of $\mathfrak{A}$ the inequality

$$\frac{|p(P) - p(P')|}{\overline{PP'}^{\alpha/2}} \leq H$$

holds. Clearly if the distance $\overline{PP'} \geq d'$ then, since $|p| \leq K_1$,

$$\frac{|p(P) - p(P')|}{\overline{PP'}^{\alpha/2}} \leq \frac{2K_1}{d'^{\alpha/2}}.$$

Therefore we need only consider the case $\overline{PP'} = s < d'$.

Let (x,y) be any point of $\mathfrak{A}$, then

$$|p(P) - p(P')| \leq |p(P) - p(x,y)| + |p(x,y) - p(P')|.$$

Introduce the domains $\Omega(P,s)$, $\Omega(P',s)$ defined above and integrate with respect to the point (x,y) over their intersection. The inequality thus obtained is

$$A\,|p(P) - p(P')| \leq \iint |p(P) - p(x,y)|\,dx\,dy + \iint |p(x,y) - p(P')|\,dx\,dy,$$

[6]Since the boundary curves have no corners we may use—instead of a quarter circle—any fraction of a circle less than $\frac{1}{2}$, i.e. any fraction bounded by two radii meeting at an angle less than π. If the boundary had corners we could use any angle less than the smallest corner angle; thus the lemma may clearly be generalized to hold for a wider class of domains.

where A is the area of this intersection. Since A is not less than $s^2/4$ we have

$$\frac{s^2}{4} \mid p(P) - p(P') \mid \leq \iint \mid p(P) - p(x,y) \mid dx\, dy + \iint \mid p(x,y) - p(P') \mid dx\, dy.$$

The first term on the right is certainly not decreased if we extend the domain of integration to all of $\Omega(P,s)$. Introducing polar coordinates (r,θ) about P we see that this first integral is not greater than

$$I_1 = \iint_{\Omega(P,s)} \left[\int_0^r \mid p_\rho(\rho,\theta) \mid d\rho \right] r\, dr\, d\theta.$$

Following the procedure of No. 2 we obtain the same estimate (2.3) derived there (for, the calculations performed there may be carried over to integration in $\Omega(P,s)$),

$$I_1 \leq \sqrt{\frac{\pi M}{2\alpha}}\, s^{2+\alpha/2}.$$

The same estimate may be obtained for the second term on the right of the inequality above. This inequality yields, therefore,

$$\frac{s^2}{4} \mid p(P) - p(P') \mid \leq \sqrt{\frac{2\pi M}{\alpha}}\, s^{2+\alpha/2}$$

or

$$\frac{\mid p(P) - p(P') \mid}{\overline{PP'}^{\alpha/2}} \leq 4\sqrt{\frac{2\pi M}{\alpha}},$$

since $\overline{PP'} = s$. The right side is clearly a constant depending only on M and α; the lemma is proved.

Remarks analogous to those at the end of No. 2 apply also to Lemma 1′. If the domain $\mathfrak{A}$ is connected and bounded it may be shown that the coefficient in the Hölder inequality satisfied by p depends only on d, M, α, and the domain $\mathfrak{A}$, and is thus independent of K_1.

3. *Hölder Continuity of Quasi-conformal and Other Mappings*

1. Functions $p(x,y)$ and $q(x,y)$, defined in a domain in the x,y-plane, define a mapping of the domain into the p,q-plane. If p and q are differentiable, the mapping behaves like an affine transformation in the neighborhood of a point, and takes circles either into ellipses or into line segments. In the former case the ratio of the major to the minor axis of the ellipse is called the "eccentricity" of the mapping at the point. If the eccentricity of the mapping at every point is uniformly bounded the mapping is said to be of "bounded eccentricity." For a mapping $p(x,y)$ $q(x,y)$ which changes the orientation the property of bounded

eccentricity may be expressed analytically by the assertion that there exists a non-negative constant k such that the derivatives of p and q satisfy everywhere the inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_y q_x - p_x q_y). \tag{3.1}$$

Mappings $p(x,y)$, $q(x,y)$ satisfying (3.1) for some constant k are also called "quasi-conformal." They have many properties similar to conformal mappings; for instance a maximum principle holds for each of the functions p, q; i.e., each function assumes its maximum and minimum on the boundary of the domain. Note in particular from the following form of (3.1)

$$\left(p_x + \frac{k}{2} q_y\right)^2 + \left(p_y - \frac{k}{2} q_x\right)^2 + \left(1 - \frac{k^2}{4}\right)(q_x^2 + q_y^2) \leq 0,$$

that if $k < 2$ the functions p, q are constant, and if $k = 2$ the functions q, p satisfy the Cauchy-Riemann equations, i.e., the mapping is conformal. Therefore, the only values of k that are of interest are $k \geq 2$.

In this section we shall consider a class of mappings somewhat more general in character, namely those satisfying an inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_y q_x - p_x q_y) + k_1 \tag{3.2}$$

where k and k_1 are non-negative constants. We shall prove that for all mappings in this class the functions $p(x,y)$ and $q(x,y)$ satisfy a Hölder inequality with constants depending on k and k_1. This implies, for $k_1 = 0$, that all functions p, q defining quasi-conformal mappings with uniformly bounded eccentricity satisfy a uniform Hölder condition. Such a Hölder inequality for quasi-conformal mappings was derived by Morrey [13] (Theorems 1 and 2 of Section 2) for one-to-one mappings. Our mappings need not be one-to-one.

In order to obtain a Hölder condition for p and q with constants in terms of k and k_1 it is easily seen, as above, that the only values of k that are of interest are $k \geq 2$. The object of this section is to prove the following

Lemma 2: Let $p(x,y)$, $q(x,y)$ be functions defined in a domain $\mathfrak{A}$ in the x,y-plane, bounded in absolute value by a constant K_1 with continuous first derivatives satisfying the inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_y q_x - p_x q_y) + k_1 \tag{3.2}$$

in $\mathfrak{A}$, where k and k_1 are non-negative constants. In any closed subdomain $\mathfrak{B}$ of $\mathfrak{A}$, the functions p and q satisfy a Hölder inequality with constants depending only on k, k_1, K_1, and the distance from the closed subdomain $\mathfrak{B}$ to the boundary of $\mathfrak{A}$.

Before proving the lemma it is of interest to consider some mappings for which Hölder inequalities may be established in all of $\mathfrak{A}$. Such cases occur when something is known concerning the values of the mapping functions on the boundary; one particular illustration of this is given by Lemma 4 of §6, 1. Other cases of interest are those where p and q admit extension to a larger domain $\mathfrak{D}$

containing the closure $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ in which (3.2) is satisfied (with possibly different constants k and k_1 depending on the original constants). Lemma 2 applied in $\mathfrak{D}$ then yields a Hölder inequality for p and q in the closed subdomain $\bar{\mathfrak{A}}$ of $\mathfrak{D}$. Extensions of this kind are possible, say by reflection, if something is known of the nature of the boundary of $\mathfrak{A}$ and of its image. To illustrate this remark we prove the following lemma, which will not be referred to in our discussion of differential equations.

Lemma 2′: Assume that $p(x,y)$, $q(x,y)$ satisfy (3.2) and map, continuously, a closed domain $\bar{\mathfrak{A}}$ bounded by a finite number of disjoint circles into a bounded multiply connected closed domain $\bar{\mathfrak{A}}'$ having the same property; assume also that each boundary circle is mapped into a boundary circle of $\bar{\mathfrak{A}}'$.[7] Then p and q satisfy a Hölder inequality in $\bar{\mathfrak{A}}$ with constants depending only on k, k_1, and the closed domains $\bar{\mathfrak{A}}$, $\bar{\mathfrak{A}}'$.

Note that the mapping is not assumed to be one-to-one or onto the whole of $\bar{\mathfrak{A}}'$.

The mapping considered in the lemma is not as special as it appears; for, by transformations of the variables x, y and the variables p, q many mappings $x, y \to p(x,y), q(x,y)$ may be reduced to this case. Under such transformations of the variables x, y or the variables p, q, inequality (3.2) is transformed into a similar inequality for the new variables, with new constants k, k_1 depending on the stretching factor introduced by the transformation. In case of a conformal transformation of either (x,y) or (p,q) variables the constant k is unchanged.

The proof of Lemma 2′ follows from Lemma 2, as indicated, by extension of the mapping functions to a larger domain $\mathfrak{D}$ by means of reflections on the boundary circles of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{A}}'$. Before defining this extension however we first map the p,q-plane one-to-one and conformally onto the p',q'-plane by means of a bilinear transformation in such a way that $\bar{\mathfrak{A}}'$ is mapped onto a closed domain $\bar{\mathfrak{A}}''$ also bounded by circles, and the outer boundary circle of $\bar{\mathfrak{A}}''$ has unit radius, is concentric with Γ (one of the inner boundary circles) and has the origin as center. Combining the mappings we have a mapping $p'(x,y)$, $q'(x,y)$ of $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{A}}''$. In $\bar{\mathfrak{A}}$, p' and q' satisfy an inequality of the form (3.2) with a different constant k_1 which is easily calculated (it depends on the original k_1 and on the closed domain $\bar{\mathfrak{A}}'$, while k remains invariant under conformal change of variables).

In order to derive a Hölder inequality for the functions p, q in $\bar{\mathfrak{A}}$ it suffices to derive such an inequality for p', q'. We derive this by extending the mapping functions p', q' to a larger domain $\mathfrak{D}$ by means of reflections in the boundary circles of $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{A}}''$. In $\mathfrak{D}$, p' and q' satisfy an inequality of the form (3.2) with again a different constant k_1 (k remaining invariant). Furthermore, in $\mathfrak{D}$, p' and q' are bounded by the inverse of the radius of the circle Γ. Application of Lemma 2 in $\mathfrak{D}$ now yields the desired Hölder inequality for p', q' in the closed subdomain $\bar{\mathfrak{A}}$.

[7] Nothing else is assumed (in particular about regularity) of the mapping of the boundary.

Clearly if $\bar{\mathfrak{A}}'$ is simply connected, i.e., bounded by one circle, the proof cannot be carried out (for, on reflection, $p^2 + q^2$ may become infinite); in fact in this case the assertion of Lemma 2′ need not be true, as a simple counterexample shows: The functions $q_n + ip_n = (x + iy)^n$, $n = 1, 2 \cdots$ map (even conformally) the unit circle $x^2 + y^2 \leq 1$ onto itself with boundary onto boundary, but do not satisfy a uniform Hölder condition in the circle.

L. Ahlfors and M. Lavrentiev[8] have derived a Hölder inequality for one-to-one quasi-conformal mappings (satisfying (3.1)) of the unit circle onto itself which preserve the origin and map boundary onto boundary. The constants of the inequality depend only on k. This result can be derived as a simple consequence of Morrey's theorem for one-to-one quasi-conformal mappings and we present a proof using Lemma 2. We formulate the result as

Lemma 2″: Let $p(x,y)$, $q(x,y)$ define a one-to-one quasi-conformal mapping (satisfying (3.1)) of the circle $x^2 + y^2 \leq 1$ onto the circle $p^2 + q^2 \leq 1$ such that the origin is mapped onto the origin and boundary onto boundary. Then p and q satisfy in the unit circle a Hölder inequality with constants depending only on k.

As in the proof of Lemma 2′ the proof involves the extension of the definition of p, q to a slightly larger circle $\mathfrak{D}$ of radius $1/\rho$ by reflection of a ring R, $\rho^2 < x^2 + y^2 \leq 1$, in the boundary circle whereupon Lemma 2 may be applied in $\mathfrak{D}$. In order to apply Lemma 2 a bound for p and q in $\mathfrak{D}$ is needed. We must first show that the points $(p(x,y), q(x,y))$ for (x,y) in R are bounded away from the origin. By the one-to-one property of the mapping its inverse $x(p,q)$, $y(p,q)$ exists and is also quasi-conformal; in fact, it satisfies an inequality of the form (3.1) in the circle $p^2 + q^2 \leq 1$ with the same constant k. We may apply Lemma 2 to this inverse mapping and conclude that there exists a circle C: $p^2 + q^2 \leq \rho'^2$, with ρ' depending only on ρ and k, such that $x^2(p,q) + y^2(p,q) \leq \rho^2$. It follows that $p^2 + q^2 > \rho'^2$ for (x,y) in R, and hence $p^2 + q^2 \leq 1/\rho'^2$ throughout $\mathfrak{D}$. Having a bound for p and q in $\mathfrak{D}$ we may apply Lemma 2 and derive a Hölder inequality for p and q in the closed subdomain $x^2 + y^2 \leq 1$.

2. Lemma 2 is a consequence of the following lemma (which is used again in §6, 2, 3 and §9, 3).

Lemma 3: Let p and q be continuous functions defined in a domain $\mathfrak{A}$ and having continuous first derivatives satisfying the inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_x q_x - p_y q_y) + k_1 \tag{3.2}$$

in $\mathfrak{A}$, where k and k_1 are non-negative constants. Assume that $|q| \leq K_1$. Let $\mathfrak{B}$ be any closed subdomain of $\mathfrak{A}$ and denote its distance from the boundary of $\mathfrak{A}$ by $2d$. Then there exist positive constants M and $\alpha < 1$ depending only on

[8]Lavrentiev derives this in §1 of "A Fundamental Theorem of the Theory of Quasi-conformal Mappings of Two-dimensional Regions," Izvestiya Akad. Nauk SSSR, Ser. Matem. 12, 1948, pp. 513–554. English version: A.M.S. Translation No. 29. Lavrentiev ascribes the result to Ahlfors who did not publish it.

k, k_1, K_1 and d such that the following inequality holds

$$\iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M,$$

where the integration is extended over any circle C_d with centre in $\mathfrak{B}$ and radius d; r denotes the distance from the centre of the circle to the point of integration.

The proof of Lemma 2 follows immediately from Lemmas 3 and 1.

The remainder of No. 2 and Nos. 3, 4, are concerned with the

Proof of Lemma 3: Consider any circle C_d with centre in $\mathfrak{B}$ and radius d and let C be a concentric circle with radius $3d/2$ (it lies in $\mathfrak{A}$) in which polar coordinates (r,θ) about the centre are introduced.

We introduce a function $\zeta(r)$ defined in the circle C with the following properties: (a) ζ is a continuous function of r alone, and is continuously differentiable; (b) ζ is identically one for $0 \leq r \leq d$ and decreases monotonically to zero as r tends to $3d/2$.[9]

We multiply (3.2) by $r^{-\alpha}\zeta^2$, where $\alpha < 1$ is a positive number to be determined later, and integrate over C, denoting $\iint_C r^{-\alpha}\zeta^2(p_x^2 + p_y^2)\, dx\, dy$ by $I_C[p]$ (and similarly for q); we obtain

$$\begin{aligned} I_C[p] + I_C[q] \leq k \iint_C r^{-\alpha}\zeta^2(p_y q_x - p_x q_y)\, dx\, dy \\ + k_1 \iint_C r^{-\alpha}\zeta^2\, dx\, dy = I. \end{aligned} \tag{3.3}$$

Our aim is to estimate the right hand side I of the inequality in the form $I \leq c(I_C[p] + I_C[q]) + \bar{c}$ where $c < 1$ and $\bar{c}$ are constants; this estimate, inserted in (3.3) would yield a bound for $I_C[p] + I_C[q]$, and hence (by the property (b) of ζ) a proof of Lemma 3.

Integration by parts of the first integral on the right (integrating the derivatives of q) yields the identity

$$k \iint_C r^{-\alpha}\zeta^2(p_y q_x - p_x q_y)\, dx\, dy = -k \iint_C (r^{-\alpha}\zeta^2)_r q(p_y r_x - p_x r_y)\, dx\, dy. \tag{3.4}$$

Here the subscript r refers to differentiation with respect to r; $r^{-\alpha}\zeta^2$ is a function of r alone. Since ζ vanishes on the boundary of C the integration by parts does not give rise to a boundary integral; this is the reason for introducing ζ. The integration by parts would certainly be valid if $p(x,y)$ had continuous second

[9] One may prove the lemma without introducing the function ζ; this would involve finding estimates for integrals $I(s) = \iint_{C_s} r^{-\alpha}(p_x^2 + p_y^2)\, dx\, dy$, over circles of radius s, from a differential inequality satisfied by $I(s)$ as a function of s. The use of the function ζ was suggested by Friedrichs in order to by-pass the differential inequality and to enable one to obtain the estimates for $I(s)$ directly.

derivatives. We may however approximate p by a sequence of functions p_n having continuous second derivatives, whose first derivatives also approximate those of p. Clearly the identity (3.4) holds for these functions p_n and letting $n \to \infty$ we see that (3.4) holds also for the function $p(x,y)$.

Replacing the first integral on the right of (3.3) by the integral on the right of (3.4), which we write as two integrals, we express I as the following sum of integrals

$$I = I_1 + I_2 + I_3, \tag{3.5}$$

where

$$I_1 = -k \iint_C 2\zeta\zeta_r r^{-a} q(p_y r_x - p_x r_y)\, dx\, dy,$$

$$I_2 = k\alpha \iint_C r^{-a-1}\zeta^2 q(p_y r_x - p_x r_y)\, dx\, dy,$$

and

$$I_3 = k_1 \iint_C r^{-a}\zeta^2\, dx\, dy.$$

The integrals I_1 and I_3 are easily estimated in terms of $I_C[p] + I_C[q]$. Consider first I_3; since $\zeta^2 \leq 1$ we have

$$I_3 \leq k_1 \iint_C r^{-a}\, dx\, dy = \frac{2\pi k_1}{2-a}\cdot\left(\frac{3d}{2}\right)^{2-a}. \tag{3.6}$$

Next,

$$I_1 \leq 2k \iint_C r^{-a} \mid q\zeta_r \mid \zeta\sqrt{p_x^2 + p_y^2}\, dx\, dy$$

$$\leq k \iint_C r^{-a}[\kappa q^2\zeta_r^2 + \kappa^{-1}\zeta^2(p_x^2 + p_y^2)]\, dx\, dy$$ [10]

where κ is any positive constant. Since $\mid q \mid \leq K_1$ we have

$$I_1 \leq k\kappa K_1^2 \iint_C r^{-a}\zeta_r^2\, dx\, dy + k\kappa^{-1} I_C[p]. \tag{3.7}$$

3. *Estimate of* I_2. We note first that the term $p_y r_x - p_x r_y$ in I_2, expressed in polar coordinates, is simply $(1/r)p_\theta$, so that the integral

$$\oint (p_y r_x - p_x r_y)\, d\theta,$$

taken around any circle r = constant, vanishes. It follows that the double

[10] This follows from the general inequality $\mid ab \mid \leq \frac{1}{2}(\kappa a^2 + \kappa^{-1}b^2)$, where κ is any positive number.

integral

$$\iint_C f(r)(p_\theta r_\theta - p_r r_\theta) r \, dr \, d\theta$$

vanishes identically for any function $f(r)$ which is a function of r alone. Thus, adding any function of r alone to q in the integral I_2 does not change the value of the integral. Let $\bar{q}(r)$ denote the value of $q(r,\theta)$ for $\theta = 0$, $\bar{q}(r) = q(r,0)$. Clearly $\bar{q}(r)$ is a function of r alone, therefore, as a consequence of the previous remarks I_2 may be written in the form

$$I_2 = k\alpha \iint_C r^{-\alpha-1}\zeta^2(q - \bar{q})(p_\theta r_\theta - p_r r_\theta) r \, dr \, d\theta, \tag{3.8}$$

which we now proceed to estimate:

$$\begin{aligned} I_2 &\leq k\alpha \iint_C r^{-\alpha}\zeta^2[r^{-1} \mid q - \bar{q} \mid \sqrt{p_r^2 + p_\theta^2}] r \, dr \, d\theta \\ &\leq k\alpha \iint_C r^{-\alpha}\zeta^2 \frac{1}{2}[r^{-2}(q - \bar{q})^2 + p_r^2 + p_\theta^2] r \, dr \, d\theta \\ &= \frac{k\alpha}{2} \int r^{-\alpha-2}\zeta^2 r \, dr \oint (q - \bar{q})^2 \, d\theta + \frac{k\alpha}{2} I_C[p]. \end{aligned}$$

Let us investigate the integral $\oint (q - \bar{q})^2 \, d\theta$ occurring above. By the definition of $\bar{q}$, the function $q - \bar{q}$, considered as a function of θ, vanishes at $\theta = 0$. We may therefore estimate the integral of the square of the function in terms of the integral of the square of its derivative:

$$\oint (q - \bar{q})^2 \, d\theta = \int_0^{2\pi} d\theta \left[\int_0^\theta q_\phi \, d\phi\right]^2 \leq \int_0^{2\pi} d\theta \left(\theta \cdot \int_0^\theta q_\phi^2 \, d\phi\right)$$

by Schwarz's inequality,

$$\leq 4\pi^2 \int_0^{2\pi} q_\theta^2 \, d\theta.$$

Thus, since $q_r^2 + q_\theta^2 = q_r^2 + (1/r^2)q_\theta^2$, we have

$$\oint (q - \bar{q})^2 \, d\theta \leq 4\pi^2 r^2 \oint (q_r^2 + q_\theta^2) \, d\theta. \tag{3.9}$$

Inserting this inequality into the last estimate for I_2 we obtain finally the estimate

$$I_2 \leq 2\pi^2 k\alpha I_C[q] + \frac{k\alpha}{2} \cdot I_C[p]. \tag{3.10}$$

4. *Completion of Proof.* We have derived the estimates for I_1, I_2, and I_3 given by (3.7), (3.10) and (3.6), respectively. Inserting their sum as an estimate

for I, into (3.3), we obtain the inequality

$$I_C[p] + I_C[q] \leq \left(k\kappa^{-1} + \frac{k\alpha}{2}\right) I_C[p] + 2\pi^2 k\alpha I_C[q]$$
$$+ k\kappa K_1^2 \iint_C r^{-\alpha}\zeta_r^2\, dx\, dy + \frac{2\pi k_1}{2-\alpha}\left(\frac{3d}{2}\right)^{2-\alpha}.$$

This inequality will enable us to estimate $I_C[p] + I_C[q]$ provided that each of the coefficients of $I_C[p]$ and $I_C[q]$, on the right, is less than unity (equals, say, 2/3). This is achieved by choosing appropriate values for κ and α which, up to now, were arbitrary. Choose $\alpha < 1$ so that

$$2\pi^2 k\alpha \leq \frac{2}{3},$$

and then choose κ so that

$$k\kappa^{-1} + \frac{k\alpha}{2} = \frac{2}{3}.$$

With κ and α thus fixed we have

$$I_C[p] + I_C[q] \leq 3k\kappa K_1^2 \iint_C r^{-\alpha}\zeta_r^2\, dx\, dy + \frac{6\pi k_1}{2-\alpha}\left(\frac{3d}{2}\right)^{2-\alpha}.$$

The terms on the right are bounded, thus we have

$$I_C[p] + I_C[q] \leq M,$$

where M is a constant depending only on k, k_1, K_1 and d.

Since $\zeta(r)$ is equal to unity for $0 \leq r \leq d$, i.e. in C_d, it follows that

$$I_C[p] = \iint_C r^{-\alpha}\zeta^2(p_x^2 + p_y^2)\, dx\, dy \geq \iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2)\, dx\, dy.$$

Thus

$$\iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2)\, dx\, dy + \iint_{C_d} r^{-\alpha}(q_x^2 + q_y^2)\, dx\, dy \leq M,$$

where M is a constant depending only on k, k_1, K_1 and d. The inequality holds for any circle C_d with radius d and center in $\mathcal{B}$.

Thus Lemma 3 is proved.

5. *Remarks.* The assumption made in Lemma 3 that the first derivatives of p and q are continuous is unnecessary. It may be shown that it is sufficient to assume that the first derivatives of p and q are measurable, that integrals of the form $I_C[p]$ and $I_C[q]$ converge, and that inequality (3.2) is satisfied almost everywhere.

We note further that the bound $|q| \leq K_1$ was used only in estimating I_1. But, for this purpose a bound for the double integral of q^2 would have sufficed. It follows, therefore, from the remarks at the end of §2, 2 that if $\mathfrak{A}$ and the closed subdomain $\mathfrak{B}$ are bounded and connected then the functions p and q of Lemma 2 satisfy a Hölder inequality in $\mathfrak{B}$ with constants which depend only on k, k_1, $\bar{K}_1$ and d, where $\bar{K}_1$ is a bound for the integral of the square of one of the functions p, q over the domain $\mathfrak{A}$.

By a refinement of the argument on page 120 (and by a somewhat different definition of $\bar{q}$) we may eliminate the factor $4\pi^2$ in (3.9) and obtain a Hölder inequality for p and q having as exponent any positive number less than $1/k$. By a somewhat different procedure, namely using "growth" of the Dirichlet integral described in §2, 2, we may obtain a Hölder inequality with exponent *equal* to $1/k$.

We note finally that since the estimates used in proving Lemmas 2 and 3 are local the lemmas may be extended to non-planar domains, say domains on a Riemann manifold.

The proof of Lemma 2 may be extended to quasi-conformal mappings in any number of variables and yields a Hölder inequality for such mappings.

4. *Proof of Theorem IV*

Let z be the given solution of

$$Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0 \tag{1.3}$$

occurring in Theorem IV. To derive the Hölder inequality satisfied by the first derivatives $p = z_x$, $q = z_y$ of z it is sufficient, in view of Lemma 2, to prove the following

Remark: If z is a solution of (1.3) satisfying conditions (i)–(iii) of Theorem IV then p and q satisfy an inequality of the form (3.2), with constants k and k_1 depending only on K, K_1 and λ of conditions (i)–(iii).

The proof of Theorem IV then follows immediately from Lemma 2 which may be applied, since p and q are bounded by K_1.

The Remark is easily proved: By the ellipticity of the equation (see (1.4)) the function C is positive, so we may divide equation (1.3) by C:

$$Ep_x + Fp_y + q_y + G = 0, \tag{4.1}$$

where $E = A/C$, $F = B/C$, $G = D/C$. From the conditions (of Theorem IV) on A, B, C, D it follows that the coefficients E, F, G are bounded in absolute value by K/λ and that for any real numbers ξ, η the inequality

$$E\xi^2 + F\xi\eta + \eta^2 \geq \frac{\lambda}{K}(\xi^2 + \eta^2) \tag{4.2}$$

holds in $\mathfrak{A}$.

Consider the pair of equations satisfied by p and q

$$Ep_x + Fp_y + q_y + G = 0, \tag{4.1}$$

and

$$p_y - q_x = 0; \tag{4.3}$$

equation (4.3) is an identity in (x,y). Multiply (4.1) by p_x, (4.3) by p_y, and add; after transposition of some terms to the right side we obtain

$$Ep_x^2 + Fp_xp_y + p_y^2 = p_yq_x - p_xq_y - Gp_x .$$

By (4.2) the left side of this equation is not less than $\lambda(p_x^2 + p_y^2)/K$ so that

$$\frac{\lambda}{K}(p_x^2 + p_y^2) \leq p_yq_x - p_xq_y - Gp_x .$$

Since $|G| \leq K/\lambda$ we see that

$$|Gp_x| \leq \frac{K}{2\lambda}(c + c^{-1}p_x^2),$$

where c is any positive number (see footnote 10 on page 119). Inserting this into the previous inequality which we multiply by K/λ we obtain the inequality

$$p_x^2 + p_y^2 \leq \frac{K}{\lambda}(p_yq_x - p_xq_y) + \frac{K^2}{2\lambda^2}(c + c^{-1}p_x^2).$$

Now choosing c so that $K^2c^{-1}/2\lambda^2$ = any number less than one, say $\frac{1}{2}$,

$$c = K^2/\lambda^2$$

we find

$$p_x^2 + p_y^2 \leq \frac{2K}{\lambda}(p_yq_x - p_xq_y) + \frac{K^4}{\lambda^4}.$$

Similarly we may show that

$$q_x^2 + q_y^2 \leq \frac{2K}{\lambda}(p_yq_x - p_xq_y) + \frac{K^4}{\lambda^4}$$

so that, on addition

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq \frac{4K}{\lambda}(p_yq_x - p_xq_y) + \frac{2K^4}{\lambda^4}, \tag{4.4}$$

which has the form of (3.2).

Remark: Following the remarks of §3, 5, it may be shown that in deriving a Hölder inequality for the first derivatives of a solution z of (1.3) it is sufficient to assume that p and q are bounded, and have measurable derivatives, that integrals of the form (2.1) converge, and that (4.1) and (4.3) are satisfied almost everywhere. In fact if $\mathfrak{A}$ and the closed subdomain $\mathfrak{B}$ are bounded and connected sets it follows from the Remark of §2, 2 and from Lemma 3 that the bound for only one of the functions (say $|q| \leq K_1$) is needed.

We note further that the system of equations (4.1) and (4.3) for p and q is elliptic. One may show in a similar manner that two functions p and q, which are solutions of a general linear elliptic system of two equations of first order, satisfy a Hölder condition whose constants may be estimated as above. This was also done by Morrey in [13].

5. *Proof of Theorem I*

We consider the differential equation

$$F(x,y,z,p,q,r,s,t) = 0 \tag{1.1}$$

in the domain $\mathfrak{D}$. Let $\mathfrak{B}$ be any closed subdomain in $\mathfrak{D}$; denote its distance from the boundary of $\mathfrak{D}$ by $2d$. Let $\mathfrak{A}$ be the open domain consisting of all points in $\mathfrak{D}$ whose distance from the boundary of $\mathfrak{D}$ is greater than d. Clearly $\mathfrak{A}$ contains $\mathfrak{B}$. Denote the closure of $\mathfrak{A}$ by $\bar{\mathfrak{A}}$.

Let (x,y) be any point of $\bar{\mathfrak{A}}$ and define the difference quotient

$$z^h(x,y) = \frac{z(x+h,y) - z(x,y)}{h}$$

where h is a positive constant less than d. Consider the differential equation

$$F(x,y,z,z_x, \cdots, z_{yy}) = 0$$

at the points $(x+h,y)$ and (x,y). Subtraction of these expressions results in the equation

$$F(x+h, y, z(x+h, y), z_x(x+h, y), \cdots, z_{yy}(x+h, y)) \\ - F(x, y, z(x, y), z_x(x, y), \cdots, z_{yy}(x, y)) = 0.$$

The left hand side may be expressed by

$$\int_0^1 \frac{d}{d\tau} F(x+\tau h, y, (1-\tau)z(x,y) + \tau z(x+h,y), \cdots, (1-\tau)z_{yy}(x,y) \\ + \tau z_{yy}(x+h,y))\, d\tau \\ = h[\tilde{F}_z + \tilde{F}_z z^h + \tilde{F}_p z_x^h + \tilde{F}_q z_y^h + \tilde{F}_r z_{xx}^h + \tilde{F}_s z_{xy}^h + \tilde{F}_t z_{yy}^h]$$

where in general $\tilde{\phi}$ represents

$$\tilde{\phi} = \int_0^1 \phi(x+\tau h, y, (1-\tau)z(x,y) \\ + \tau z(x+h,y), \cdots, (1-\tau)z_{yy}(x,y) + \tau z_{yy}(x+h,y))\, d\tau.$$

The difference quotient z^h evidently satisfies the linear differential equation

$$\tilde{F}_r z_{xx}^h + \tilde{F}_s z_{xy}^h + \tilde{F}_t z_{yy}^h + \tilde{F}_p z_x^h + \tilde{F}_q z_y^h + \tilde{F}_z z^h + \tilde{F}_z = 0.$$

Since the first derivatives of F are continuous, and since $\bar{\mathfrak{A}}$ is closed, we may be sure that for h sufficiently small the functions $\tilde{F}_r, \tilde{F}_s, \cdots, \tilde{F}_z$ differ but slightly

from F_r, F_s, $\cdots$, F_z and hence are bounded by $2K$ (K being the given bound for F_t, F_s, $\cdots$, F_z).

Furthermore, we may be sure that for all real numbers ξ, η,

$$\tilde{F}_r\xi^2 + \tilde{F}_s\xi\eta + \tilde{F}_t\eta^2 \geq \frac{\lambda}{2}(\xi^2 + \eta^2)$$

for all (x,y) in $\bar{\mathfrak{A}}$. Since z_x^h and z_y^h tend to z_x and z_y as $h \to 0$ they are bounded by $2K_1$ for h sufficiently small. These statements follow from assumptions (i), (ii) and (iii) of Theorem I.

The equation which z^h satisfies may be written in the form

$$\tilde{F}_r z_{xx}^h + \tilde{F}_s z_{xy}^h + \tilde{F}_t z_{yy}^h + \tilde{D} = 0 \tag{5.1}$$

where $\tilde{D} = \tilde{F}_p z_x^h + \tilde{F}_q z^h + \tilde{F}_s z^h + \tilde{F}_z$. Note that for h sufficiently small

(i) $\tilde{F}_r$, $\tilde{F}_s$, $\tilde{F}_t$, $\tilde{D}$, are bounded by a constant depending on K and K_1,

(ii) $z^h(x,y)$ has continuous first and second derivatives, the first derivatives being bounded by $2K_1$ in $\mathfrak{A}$.

(iii) For all real numbers ξ, η the inequality

$$\tilde{F}_r\xi^2 + \tilde{F}_s\xi\eta + \tilde{F}_t\eta^2 \geq \frac{\lambda}{2}(\xi^2 + \eta^2)$$

holds for all (x,y) in $\mathfrak{A}$.

Theorem IV proved in §4 may therefore be applied to equation (5.1) in the open domain $\mathfrak{A}$, and we find that z_x^h and z_y^h satisfy a Hölder condition in the closed subdomain $\mathfrak{B}$, with constants depending only on K, K_1, λ and d. That is, this Hölder condition is independent of the value of h. Letting $h \to 0$ we see that the limit functions $\lim_{h\to 0} z_x^h$ and $\lim_{h\to 0} z_y^h$ satisfy the same Hölder conditions; these limit functions are simply z_x and z_y.

Similarly we may show that z_{yy} satisfies the same Hölder condition in $\mathfrak{B}$ and Theorem I is proved.

As mentioned at the end of the previous section, the results of Theorem IV may be established for a system of elliptic equations of first order in two functions $p(x,y)$ and $q(x,y)$. Theorem I may therefore be extended to a nonlinear system in two functions $p(x,y)$, $q(x,y)$. The statement is the following:

If $p(x,y)$ and $q(x,y)$ have continuous first derivatives and satisfy a general (nonlinear) elliptic system of first order then the first derivatives of p and q satisfy Hölder conditions in any closed subdomain of the original domain.

The author has generalized this result to the most general elliptic system of equations for functions of two independent variables. This result will appear in a later communication.

Another generalization of Theorem I is mentioned in footnote 3, page 104.

6. *A Sharp Form of Theorem IV*

1. In this section we shall modify Theorem IV by making additional assumptions which enable us to derive a Hölder inequality for the first derivatives

of the given solution $z(x,y)$ of (1.3) in the whole domain $\mathfrak{a}$. We consider functions $z(x,y)$ which are defined in the closure $\bar{\mathfrak{a}}$ of a domain $\mathfrak{a}$ of type L_2 (see §2, 1) and satisfy the condition:

(A) z and its first and second derivatives are continuous in $\bar{\mathfrak{a}}$, and the boundary values of z, regarded as functions of arc length, have continuous first and second derivatives—the second derivatives being bounded in absolute value by a constant K_2.

We now restate Theorem IV in a sharper form as

Theorem V: Let $z(x,y)$ be defined in $\bar{\mathfrak{a}}$ and satisfy the elliptic differential equation

$$Az_{xx} + Bz_{xy} + Cz_{yy} + D = 0 \tag{1.3}$$

in $\mathfrak{a}$. Assume that conditions (i)–(iii) of Theorem IV are satisfied and assume further that z satisfies condition (A).

Conclusion: The first derivatives of z satisfy a Hölder condition in $\mathfrak{a}$ with coefficient and exponent depending only on K, K_1, K_2, λ (of conditions (i)–(iii) and (A)) and the domain $\mathfrak{a}$.

We prove, really, a more general statement from which Theorem V follows. This is a generalization of Lemma 2 of §3.

Lemma 4: Let $z(x,y)$ satisfy condition (A), and assume that the first derivatives p, q of z are bounded in absolute value by a constant K_1 and satisfy the inequality (3.2)

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_y q_x - p_x q_y) + k_1 ,$$

in $\mathfrak{a}$, with k and k_1 non-negative constants.

Conclusion: p and q satisfy a Hölder condition in $\mathfrak{a}$ with coefficient and exponent depending only on k, k_1, K_1, K_2 and the domain $\mathfrak{a}$.

That Theorem V is an immediate consequence of Lemma 4 may be seen with the aid of the Remark at the beginning of §4.

Lemma 4 may be strengthened: if, instead of using the bound K_2 for the second derivatives of the boundary values, we indroduce a bound K_3 for the integral of the squares of these second derivatives—with respect to arc length along the boundary—then, with the aid of the remarks at the end of §7, we may obtain a Hölder inequality for p and q with constants depending only on k, k_1, K_1, K_3 and $\mathfrak{a}$. This stronger result will not be used.

Lemma 4 is related to certain results obtained by J. Leray in a paper [7] concerning nonlinear elliptic equations of second order. In Sections 5, 6, 10–12 of [7] Leray derives an estimate of the *modulus of continuity* of the first derivatives of a solution of a nonlinear elliptic equation of the form (1.1)—making use of an inequality of the form (3.2) for the derivatives of the solution and of additional properties of the differential equation. Lemma 4, however, yields an estimate for the *Hölder continuity* of these first derivatives.

2. *Outline.* Lemma 4 will be proved by generalizing the arguments used in §3 to prove Lemma 2. There, in proving first Lemma 3 (§3, 2) we derived estimates for integrals of the form

$$\iint_C r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \tag{6.1}$$

(see (2.1)) over circles C lying entirely inside the domain. Lemma 1 implied that p and q satisfy a Hölder condition in a subdomain of $\mathfrak{A}$. In the present discussion we shall derive estimates for such integrals over the intersection of $\mathfrak{A}$ with all circles C having some fixed radius and centre in $\mathfrak{A}$. The derived Hölder inequalities will then follow from Lemma 1'.

The estimates for integrals over circles lying entirely inside $\mathfrak{A}$, away from the boundary, are already furnished by Lemma 3. The proof of Lemma 3 is to be generalized to yield estimates for integrals over circles which intersect the boundary of $\mathfrak{A}$ (the integrals extend only over the part of the circles lying in $\mathfrak{A}$). In order to motivate the discussion below, imagine the argument given in §3, 2–4 for the proof of Lemma 3, applied now to a circle C, which intersects the boundary of $\mathfrak{A}$; let us see how far it may be extended. As we shall see, it may be carried over completely, with minor adjustments, *except in one particular.* It involves an integration by parts (see (3.4)), which in §3 yielded no boundary integral, but which here gives rise to an integral on the boundary of $\mathfrak{A}$; in order to carry through the rest of the argument it is necessary to obtain estimates for this boundary integral. Upon examination of the terms in the integral it is seen that the functions $p = z_x$, $q = z_y$ occur in the form $q(p_x\, dx + p_y\, dy) = qp_s d_s$, where p_s is the derivative of p with respect to arc length s on the boundary; it is apparent that this integral may be estimated, provided that a bound for $|p_s|$ on the boundary of $\mathfrak{A}$ is known ($|q|$ is bounded by K_1). The function $p_s = (z_x)_s$ is a combination of second derivatives of the function z; of these a bound K_2 is known only for the second derivative of z along the boundary with respect to s (by condition (A)). Thus a bound for p_s is known (leading to an estimate for the boundary integral) only if, on the boundary, p is the derivative of z with respect to s, i.e., *only if the part of the boundary which intersects C is a straight segment parallel to the x-axis.*

It now becomes clear how we should proceed in order to obtain the estimates for the integrals (6.1) over circles C which intersect the boundary of $\mathfrak{A}$. First we map the boundary (at least locally) into a straight segment Γ—this being achieved by a local transformation of variables—then we follow the argument described above, to obtain estimates for the integrals over the intersections of the domain with circles C which intersect the straight segment Γ. On reintroduction of the original variables these estimates yield the desired estimates for integrals over circles intersecting the boundary of $\mathfrak{A}$.

In order to carry out this procedure we formulate first the result which one obtains on applying the arguments of §3, 2–4, used in the proof of Lemma 3, to

a circle which intersects the boundary of $\mathfrak{A}$ in a straight segment. Its proof, which is somewhat lengthy, is postponed to §7.

Lemma 3′: Let $p(x,y)$, $q(x,y)$ be defined in a domain $\mathfrak{A}$ in the x,y-plane and satisfy all the conditions of Lemma 3. Assume that part of the boundary of $\mathfrak{A}$ consists of a straight segment Γ, which we may suppose to be on the x-axis, and that p, q and their derivatives of first order may be defined on Γ so that they are continuous in $\mathfrak{A} + \Gamma$. Assume finally that on Γ the condition

$$| p_x | \leq K_2$$

is satisfied, where K_2 is some positive constant.

Consider a point P in $\mathfrak{A}$ whose distance from any boundary point of $\mathfrak{A}$ not on Γ is not less than some positive constant $2c$. Denote by C_c the intersection of $\mathfrak{A}$ with a circle having centre P and radius c.

Conclusion: There exist positive constants M and $\alpha < 1$, depending only on k, k_1, K_1, K_2 and c such that the inequality

$$\iint_{C_c} r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M$$

holds.

3. *Proof of Lemma* 4. With the aid of Lemma 3′ (which is proved in §7) we proceed now with the details of the outline for the proof of Lemma 4. The discussion is mainly a technical one, no new ideas are involved, but it is presented at length in order to make clear how a similar procedure, described in §9, may be carried out.

In order to prove Lemma 4 it is sufficient to show that there exist positive numbers d', c', M' and $\alpha' < 1$ depending only on k, k_1, K_1, K_2 and the domain $\mathfrak{A}$, such that

$$\iint_{C_{c'}} r^{-\alpha'}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M', \tag{6.3}$$

holds, where $C_{c'}$ denotes the intersection of $\mathfrak{A}$ with a circle of radius c', having as centre any point P whose distance from the boundary of $\mathfrak{A}$ is less than $2d'$. For, if such numbers have been found, consider the closed subdomain $\mathfrak{B}$ consisting of all points of $\mathfrak{A}$ whose distance from the boundary of $\mathfrak{A}$ is not less than $2d'$. All the conditions of Lemma 3 are satisfied; applying the lemma to the subdomain $\mathfrak{B}$ we conclude that there exist positive numbers M_1 and $\alpha_1 < 1$ such that the inequality

$$\iint_{C_{d'}} r^{-\alpha_1}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M_1$$

holds, where $C_{d'}$ is any circle with centre in $\mathfrak{B}$ and radius d'. Setting now

$$\alpha = \min(\alpha', \alpha_1)$$
$$M = \max(M', M_1)$$
$$d = \min(c', d', 1),$$

we may combine the last inequality with (6.3) and conclude that the inequality

$$\iint_{C_d} r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M \tag{6.4}$$

holds, where C_d represents the intersection of $\mathfrak{A}$ with *any* circle having centre in $\mathfrak{A}$ and radius d. The conclusion of Lemma 4 now follows by use of Lemma 1′.

With the aid of Lemma 3′ we proceed to find the numbers d', c', M' and α'. We find first the constant d' which is determined by the domain $\mathfrak{A}$ alone. Let Q be a boundary point of $\mathfrak{A}$. Since $\mathfrak{A}$ is of type L_2 (see §2, 1) the boundary curve containing this point may be represented by an equation of the form $y = f(x)$ in a neighborhood of Q, and we may introduce

$$\xi = x, \qquad \eta = y - f(x) \tag{6.5}$$

as new independent variables in a neighborhood of Q. There exists then a circle with centre Q having the properties: (a) it contains an arc of the boundary curve containing Q and no other boundary points of $\mathfrak{A}$, (b) the transformation (6.5) maps this circle in a one-to-one way onto a domain in the ξ,η-plane. Clearly the arc of the boundary curve which contains Q and lies in the circle is mapped onto a segment of the line $\eta = 0$. Thus, if the image of the part of the circle lying in $\mathfrak{A}$ is denoted by Ω, Ω has a segment on the ξ-axis as part of its boundary. In the circle about Q the function $f(x)$ and hence the functions $\xi(x,y)$, $\eta(x,y)$ have continuous second derivatives. Introduce $\overline{K}$, an upper bound for the derivatives of ξ and η in the circle, so that ξ and η satisfy the inequalities

$$|\xi_x|,\ |\eta_x|,\ \cdots,\ |\eta_{yy}| \leq \overline{K}. \tag{6.6}$$

Such a circle may be drawn about every boundary point Q, and since the boundary of $\mathfrak{A}$ consists of a finite number of closed curves (having finite length) we conclude that there exist positive constants d' and $\overline{K}$ such that every circle with centre on the boundary of $\mathfrak{A}$ and radius $4d'$ satisfies conditions (a) and (b) (of the previous paragraph), and such that new variables ξ, η introduced in these circles (as described by (6.5)) satisfy (6.6); d' is thus determined.

Before proceeding to determine the constants c', M' and α' we note that lengths are not stretched too much under the mapping of the circles (of radius $4d'$) about boundary points, described by (6.5). Namely, it is easily seen that there exists a positive constant κ, depending only on $\overline{K}$, such that the distance l between any two points of such a circle and the distance l' between the image points under the mapping satisfy the inequality

$$\kappa \leq \frac{l}{l'} \leq \kappa^{-1}. \tag{6.7}$$

To find the constants c', M' and α' let P be any point of $\mathfrak{A}$ whose distance from the boundary of $\mathfrak{A}$ is less than $2d'$, and let Q be a boundary point of $\mathfrak{A}$ nearest to P. With Q as centre draw a circle of radius $4d'$. From the definition of d' it follows that new variables ξ,η may be introduced, as described above, mapping the circle onto a domain in the ξ,η-plane. As before, denote by Ω the

image of the part of the circle lying in $\mathfrak{A}$; Ω has a segment on $\eta = 0$ as part of its boundary. Since the distance of P from the circumference of the circle about Q is not less than $2d'$ it follows from (6.7) that the distance of P', the image of P, from any boundary point of Ω *not on* $\eta = 0$ is not less than $2\kappa d'$. Set $c = \kappa d'$.

In Ω the function z of Lemma 4 may be regarded as a function of ξ, η, which we denote by $z'(\xi,\eta)$. What properties does the function z' have as a consequence of the assumptions of Lemma 4? First, it is easily seen from these assumptions, and from (6.6), that the derivatives of z', $p' = z'_\xi$, $q' = z'_\eta$, are bounded in absolute value by a constant K'_1 and satisfy an inequality

$$p'^2_\xi + p'^2_\eta + q'^2_\xi + q'^2_\eta \leq k'(p'_\eta q'_\xi - p'_\xi q'_\eta) + k_1,$$

where K'_1, k' and k'_1 are constants depending on K_1, k, k_1 and $\overline{K}$. Furthermore, it is seen from condition (A), assumed in Lemma 4, that on the segment of $\eta = 0$ which belongs to the boundary of Ω, the inequality

$$|p'_\xi| = |z'_{\xi\xi}| \leq K'_2$$

holds, where K'_2 is a constant depending on K_2 and $\overline{K}$.

We note, finally, as a consequence of (6.6) and of the boundedness of p' and q', that at corresponding points (x,y), (ξ,η) the inequality

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq \kappa_1(p'^2_\xi + p'^2_\eta + q'^2_\xi + q'^2_\eta + 1) \tag{6.8}$$

holds, where κ_1 is a constant depending only on K_1 and $\overline{K}$ ((6.8) is used later).

From the properties described above it is clear that the functions p' and q', satisfy in Ω all the conditions of Lemma 3'. The domain $\mathfrak{A}$ and the point P in the lemma are replaced by Ω and P', and the constants k, k_1, K_1, K_2 and c of the lemma are replaced by k', k'_1, K'_1, K'_2 and $c = \kappa d'$. From the conclusion of Lemma 3' we infer that there exist positive constants $\overline{M}$ and $\alpha' < 1$ depending only on k', k'_1, K'_1, K'_2 and c, such that the inequality

$$\iint_{C'_c} \rho^{-\alpha'}(p'^2_\xi + p'^2_\eta + q'^2_\xi + q'^2_\eta)\, d\xi\, d\eta \leq \overline{M} \tag{6.9}$$

holds, where C'_c is the intersection of Ω with a circle of radius c about P', and ρ denotes the distance from the point of integration to P'.

On reintroduction of the variables x,y the integral over C'_c may be considered as an integral over a domain in the x,y-plane (the Jacobian of the transformation $\xi(x,y)$, $\eta(x,y)$ is unity). From (6.7) it follows that this domain contains the intersection of $\mathfrak{A}$ with a circle having centre P and radius

$$\kappa c = \kappa^2 d'.$$

Set $c' = \kappa^2 d'$—this is the desired constant c'—and denote the intersection of $\mathfrak{A}$ with this circle by $C_{c'}$. It follows further, from (6.7), that if r is the distance of a point of $C_{c'}$ from P, and ρ the distance of its image from P', then

$$r \geq \kappa\rho.$$

From these remarks we infer that

$$\iint_{C_c'} r^{-\alpha'}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq \iint_{C'_c} \kappa^{-\alpha'}\rho^{-\alpha'}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, d\xi\, d\eta$$

$$\leq \kappa^{-\alpha'} \iint_{C'_c} \rho^{-\alpha'}\kappa_1(p_\xi'^2 + p_\eta'^2 + q_\xi'^2 + q_\eta'^2 + 1)\, d\xi\, d\eta,$$

in virtue of (6.8),

$$\leq \kappa^{-\alpha'}\kappa_1\left(\overline{M} + \iint_{C'_c} \rho^{-\alpha'}\, d\xi\, d\eta\right)$$

in virtue of (6.9),

$$\leq a \text{ constant } M',$$

so that (6.5) is proved.

We have found the constants d', c', M' and $\alpha' < 1$, depending only on k, k_1, K_1, K_2 and the domain $\mathfrak{A}$, and proved that with these constants the inequality (6.5) holds. This completes the proof of Lemma 4.

7. *A Special form of Lemma 3.*

This section is devoted to a proof of Lemma 3′, which was used in §6, 3 to prove Theorem V.

Lemma 3′: Let $p(x,y)$, $q(x,y)$ be defined in a domain $\mathfrak{A}$ and have continuous first derivatives satisfying the inequality (3.2) or

$$p_x^2 + p_y^2 + q_x^2 + q_y^2 \leq k(p_y q_x - p_x q_y) + k_1 \tag{7.1}$$

in $\mathfrak{A}$, where k and k_1 are non-negative constants. Assume that part of the boundary of $\mathfrak{A}$ consists of a straight segment Γ, which we may suppose to be on the x-axis, and assume that p,q and their derivatives of first order may be defined on Γ so that they are continuous in $\mathfrak{A} + \Gamma$. Assume finally that $|q| \leq K_1$ and that on Γ

$$|p_x| \leq K_2 \tag{7.2}$$

where K_2 is some positive constant.

Consider a point P in $\mathfrak{A}$ whose distance from any boundary point of $\mathfrak{A}$ not on Γ is not less than some positive constant $2c$. Denote by C_c the intersection of $\mathfrak{A}$ with a circle having centre P and radius c.

Conclusion: There exist positive constants M and $\alpha < 1$ depending only on k, k_1, K_1, K_2 and c such that

$$\iint_{C_c} r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2)\, dx\, dy \leq M.$$

The proof proceeds in a manner similar to that of Lemma 3 in §3, 2–4; rather than rewrite all the details we present merely the modifications due to

the additional assumptions, and refer whenever possible to the calculations given there.

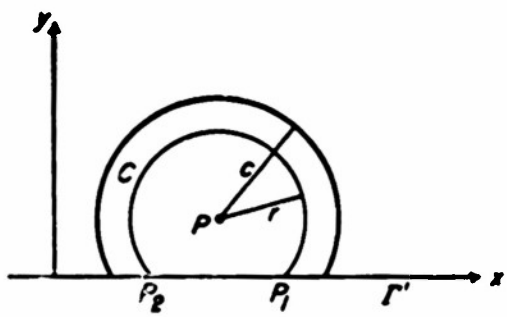
Proof: Denote by C the intersection of $\mathfrak{A}$ with a circle having centre P and radius $2c$. If the distance from P to Γ is less than $2c$, which is the case of most interest to us, the boundary of C consists of a circular arc and a straight segment Γ'. As in §3, 2, we define in C a function $\zeta(x,y)$ which depends on the polar coordinate r (distance from P) alone, is identically one for $0 \leq r \leq c$ and decreases monotonically to zero as r tends to $2c$. We follow the procedure of that section; multiply (7.1) by $r^{-\alpha}\zeta^2$ and integrate over C to obtain (3.3). Here $\alpha < 1$ is a positive constant to be determined later. Then perform the integration by parts as indicated there. This gives rise to a boundary integral over Γ', since ζ does not vanish along Γ'; therefore the expression (3.5) for I (the right side of (3.3)) is modified by the addition of this boundary integral:

$$I = I_1 + I_2 + I_3 + I_4 \tag{7.3}$$

where I_1, I_2 and I_3 are the integrals defined on page 119 (integrated over C), and I_4 is the boundary integral

$$\begin{aligned} I_4 &= k \int_{\Gamma'} r^{-\alpha}\zeta^2 q(p_x\, dx + p_y\, dy) \\ &= k \int_{\Gamma'} r^{-\alpha}\zeta^2 q p_x\, dx, \end{aligned}$$

since Γ' is a segment on the x-axis.



Region of Integration

FIGURE 1

We must now obtain estimates for the integrals I_1 to I_4 in terms of $I_c[p] + I_c[q]$. The integral I_1 is bounded as in §3, 2 by (3.7), in terms of an arbitrary constant κ, while a bound for I_3 is given by (3.6). The integral I_4 is easily estimated, using (7.2) and the inequality $|q| \leq K_1$,

$$\begin{aligned} I_4 &\leq k \int_{\Gamma'} r^{-\alpha}\zeta^2 \,|\, qp_x \,|\, dx \\ &\leq kK_1K_2 \int_{\Gamma'} r^{-\alpha}\, dx \end{aligned}$$

since $\zeta^2 \leq 1$, finally

$$I_4 \leq 2kK_1K_2 \int_0^{2c} |x|^{-\alpha}\, dx = \frac{2kK_1K_2}{1-\alpha}(2c)^{1-\alpha}. \tag{7.4}$$

We must modify the argument for estimating I_2 given in §3, 3. Consider

$$I_2 = k\alpha \iint_C r^{-\alpha-1}\zeta^2 q(p_y r_x - p_x r_y) r\, dr\, d\theta$$

where (r,θ) are polar coordinates about P. Since $p_y r_x - p_x r_y = 1/r\, p_\theta$ we note that the integral

$$\bar{p}(r) = \int_C (p_y r_x - p_x r_y)\, d\theta$$

—taken around the arc of the circle r = constant, which lies in C and has end points P_1, P_2 (see sketch)—is given by

$$\bar{p}(r) = \frac{1}{r}(p(P_2) - p(P_1)).$$

Introducing the function $\bar{q}(r) = q(r,0)$, as in §3, 3, we subtract and add $\bar{q}(r)$ to $q(r,\theta)$ in the integrand in I_2 and write the integral as a sum of two integrals

$$I_2 = I_2' + I_2'',$$

with

$$I_2' = k\alpha \iint_C r^{-\alpha-1}\zeta^2(q - \bar{q})(p_y r_x - p_x r_y) r\, dr\, d\theta,$$

and

$$I_2'' = k\alpha \iint_C r^{-\alpha-1}\zeta^2 \bar{q}(p_y r_x - p_x r_y) r\, dr\, d\theta$$

$$= k\alpha \int_0^{2c} r^{-\alpha}\zeta^2 \bar{q}\bar{p}\, dr.$$

The integral I_2' is similar to the expression (3.8) and may be estimated in a similar manner, so that the estimate (3.10) holds for I_2'.

To estimate I_2'' we observe that the term

$$\bar{p}(r) = \frac{1}{r}(p(P_2) - p(P_1)),$$

which occurs in the integrand, may be bounded by

$$|\bar{p}(r)| \leq 2K_2.$$

This follows from (7.2), using the theorem of the mean, and from the fact that the distance between the points P_1 and P_2 (which lie on a circle of radius r) is not greater than $2r$. Since $|q| \leq K_1$ and $\zeta^2 \leq 1$ it follows that

$$I_2'' \leq 2k\alpha K_1K_2 \int_0^{2c} r^{-\alpha}\, dr = \frac{2k\alpha K_1K_2}{1-\alpha}(2c)^{1-\alpha}.$$

Adding this estimate for I_2'' to the estimate (3.10) which holds for I_2' we find

$$I_2 \leq 2\pi^2 k\alpha I_c[q] + \frac{1}{2} k\alpha I_c[p] + \frac{2\alpha}{1-\alpha} kK_1K_2(2c)^{1-\alpha}.$$

Combination of this estimate with (3.7) for I_1, (3.6) for I_3 and (7.4) for I_4 yields a bound for I (given by (7.3) which, when inserted into (3.3), yields

$$I_c[p] + I_c[q] \leq A(\kappa,\alpha)(I_c[p] + I_c[q]) + B(\kappa,\alpha)$$

where $A(\kappa,\alpha)$ and $B(\kappa,\alpha)$ are constants depending on κ and α (which up to now are arbitrary) and also upon k, k_1, K_1, K_2 and c. As in §3, 4, we may choose α and κ so that $A(\kappa,\alpha)$ is less than unity—say 2/3. Then

$$I_c[p] + I_c[q] \leq 3B(\kappa,\alpha).$$

Since $\zeta(r)$ is equal to unity for $0 \leq r \leq c$ it follows that

$$I_c[p] = \iint_C r^{-\alpha}\zeta^2(p_x^2 + p_y^2)\,dx\,dy \geq \iint_{C_c} r^{-\alpha}(p_x^2 + p_y^2)\,dx\,dy,$$

where C_c is the intersection of $\mathfrak{A}$ with a circle with centre P and radius c. Thus from the estimate for $I_c[p] + I_c[q]$ we find

$$\iint_{C_c} r^{-\alpha}(p_x^2 + p_y^2 + q_x^2 + q_y^2) \leq M,$$

where M and $\alpha < 1$ are positive constants depending only on k, k_1, K_1, K_2 and c; Lemma 3′ is proved.

Remark: In this proof the bound M depends on the bound K_2 for $|p_x|$ on Γ. By modifying the proof slightly it is possible to show that M depends merely on K_3—in addition to k, k_1, K_1, c –where K_3 is a bound for the integral of p_x^2 along Γ.

8. *Quasi-linear Elliptic Differential Equations*

1. *Outline.* An application of Theorem V to quasi-linear elliptic equations will now be presented. We will consider an elliptic equation of the form

$$A(x, y, z, z_x, z_y)z_{xx} + B(x, y, z, z_x, z_y)z_{xy} + C(x, y, z, z_x, z_y)z_{yy} = 0 \tag{8.1}$$

in a convex domain, and prove the existence of a solution taking on given boundary values. The existence of a solution of this boundary value problem was first derived by J. Leray and J. Schauder [9] as an application of their theorems concerning the degree of a mapping in Banach space. They assumed that the coefficients of the equation are twice differentiable with respect to all arguments. Previously similar existence theorems had been derived—but under additional assumptions. Schauder proved the existence of a solution under the assumption that the problem of the corresponding general inhomo-

geneous equation, with an arbitrary given function of x and y on the right side, and with arbitrary prescribed boundary values, has at most one solution [18]. Much earlier S. Bernstein [2] showed the existence of a solution under the assumption that z does not occur in the coefficients (they are functions of x, y, z_x, z_y).

The following approach is used here, as well as by Leray and Schauder: We solve the linear equation[11]

$$A(x, y, z, z_x, z_y)Z_{xx} + B(x, y, z, z_x, z_y)Z_{xy} + C(x, y, z, z_x, z_y)Z_{yy} = 0 \tag{8.2}$$

for a function $Z(x,y)$ which takes on the prescribed boundary values ϕ and which corresponds to every function $z(x,y)$ belonging to a suitably defined class of functions. The function Z defines a transformation $Z(z)$ of the class of functions z. This class of functions belongs to a Banach space and is mapped—under the transformation $Z[z]$—back into the Banach space. The problem of showing the existence of a solution assuming the given boundary values is thus transformed to that of finding a fixed point of this transformation (or mapping).

Leray and Schauder carried this out by looking for zeros of the mapping $z - Z[z]$. They studied the degree of this mapping at the origin (in the Banach space) by embedding it in a one-parameter family of mappings for which the degree of mapping is invariant. This family was constructed by solving the above equation for a function Z_k which takes on the values $k\phi$ on the boundary, k being a parameter which is allowed to vary from zero to one. For $k = 0$ the solution is $Z \equiv 0$ and the mapping $z - Z_0[z]$ reduces to the identity—for which the degree is unity. Thus the degree of mapping is unity for all k, in particular for $k = 1$, and therefore the equation

$$z - Z[z] = 0$$

has a solution. To make sure that the degree of mapping does not change as k varies it was essential to know that no solutions of the equations $z - Z_k[z] = 0$ occur on the boundary of the class of functions considered (in the Banach space). Thus certain a priori bounds for solutions of the equations $z - Z_k[z] = 0$ had to be established.

Our method of finding a fixed point of the transformation $Z[z]$ is to use a fixed point theorem, due to Schauder [17], concerning completely continuous transformations in Banach space. (A transformation of a Banach space into itself is said to be completely continuous if it maps bounded sets into compact sets). The theorem states: *Let T be a completely continuous transformation defined in a closed convex set in a Banach space, and suppose that T is continuous and maps this set into itself. Then the transformation has a fixed point.*

The proof of the existence of a solution of (8.1) as given here and in [9], is based on (a) the theory of linear elliptic differential equations—for, in order to

[11]In this case the solution of the boundary value problem is unique. Whether it is also unique for the whole general case (8.1) is not known to me.

define the transformation $Z[z]$ the linear equation (8.2) must be solved for Z—and (*b*) a priori estimates for the solutions z' of (8.1) and Z of (8.2). The distinguishing feature of our discussion, as contrasted to that of Leray and Schauder, is the nature of these a priori estimates. They enable us to choose a set of functions z which will be mapped into itself under the transformation $Z[z]$—and hence to use Schauder's fixed point theorem.

Our procedure for obtaining a priori bounds is based entirely on statements concerning *linear* elliptic equations. We remark first that if z' is a solution of (8.1) then (8.1) may be considered as a linear elliptic equation for z', with known functions of x and y as coefficients (once the values of z' and its derivatives have been inserted into these coefficients). From this fact it is not difficult, as we show in No. 3, to derive bounds for z' and its derivatives of first order. In order to obtain estimates for the *second* derivatives of z' one must know more about the coefficients of the linear elliptic equation it satisfies; for instance, if these coefficients satisfy a Hölder condition then estimates for the second derivatives may be obtained. But these coefficients involve the *first* derivatives of the function z', and thus in order to calculate their Hölder continuity one must first estimate the Hölder continuity of these first derivatives. Just such estimates are obtained if we apply Theorem V to the equation (still regarded as linear) for z'; knowing these estimates we may then calculate estimates for the second derivatives of z'.

Leray and Schauder used estimates for the second derivatives of a solution z' derived earlier by Schauder in Section 4 of [18]. His procedure for obtaining these estimates was rather involved and made use of the "auxiliary function" introduced by S. Bernstein [2] (Schauder referred to pages 119–125 in [2]).

More explicitly, the procedure we shall employ for deriving a priori estimates for solutions Z of (8.2) and for solutions z' of (8.1) is the following: We consider a solution $Z[z]$ of (8.2) corresponding to some function z and seek a priori estimates for the function Z. The estimates that can be obtained depend, of course, on the assumptions on the function z. Assuming at first as little as possible we derive a priori bounds for Z and its derivatives of first order. Then we suppose that these estimates hold for z and derive, using Theorem V, an a priori Hölder inequality for the first derivatives of Z. Assuming, in turn, that these new estimates hold for z we derive still stronger estimates for Z—and so on, continuing this iteration process. Note that at each stage of this process only statements for solutions Z of *linear* equations are employed. If z' is a solution of (8.1) and we take $Z = z = z'$ this process yields the desired a priori estimates for the solution z'.

We remark that the class of functions z to which the Schauder fixed point theorem will be applied will consist of those functions satisfying the a priori estimates obtained in the first two steps of the iteration process described above; these steps are carried out in Nos. 3, 4.

C. B. Morrey attempted (with the aid of Theorem IV, (see [13] Theorem I, p. 164)) to show the existence of a fixed point of the transformation $Z[z]$ by

an iteration procedure. Starting with a function $z_1(x,y)$ he defined, recursively, the functions

$$z_n = Z[z_{n-1}] \qquad n = 2, 3, \cdots .$$

He showed that one can select a subsequence of the z_n which will converge to a function $z(x,y)$ having continuous first and second derivatives. He then stated that the function $z(x,y)$ is a solution of the differential equation; but this need not be the case.

The existence theorem to be proved—for the boundary value problem for (8.1)—is formulated as Theorem VI. The first two steps of our iteration process for deriving a priori estimates for solutions Z and z' of (8.2) and (8.1) are carried out in Nos. 3, 4. Bounds for these solutions and their derivatives of first order are obtained, and a Hölder inequality for these derivatives is derived with the aid of Theorem V. These results are then employed in No. 5 in introducing the appropriate class of functions z for which the mapping $Z[z]$ is defined and which is mapped into itself by $Z[z]$. The complete a priori estimates for solutions of (8.1) are also derived in No. 5. Finally the proof of Theorem VI is given in No. 6.

In No. 7 the existence of a solution of the boundary value problem is proved under weaker assumptions concerning the given boundary values.

This entire section is completely independent of the rest of the paper, except for a reference to Theorem V. The boundary value problem for (8.1) may be solved however without the use of Theorem V, which is a sharp form of Theorem IV, but using Theorem IV itself. This is indicated in No. 8. The solution so obtained has continuous derivatives of second order in $\mathfrak{A}$, but these derivatives need not be continuous in $\bar{\mathfrak{A}}$.

2. *Precise Formulation of Problem.* Before stating the existence theorem in precise form we introduce several Banach spaces which play a fundamental role in our discussion.

We shall always consider functions defined in the closure of a fixed convex domain $\mathfrak{A}$ which is bounded by a curve Γ of finite length represented by

$$x = x(s), \qquad y = y(s),$$

and shall assume that the functions $x(s)$, $y(s)$ of arc length s have continuous derivatives up to the third order, and that Γ has positive curvature everywhere.

Denote by C_m the class of real functions $z(x,y)$ having continuous partial derivatives up to order m where m is a non-negative integer (these are to be continuous in the closure $\bar{\mathfrak{A}}$ of $\mathfrak{A}$) and by $C_{m+\alpha}$ the sub-class of functions in C_m whose derivatives satisfy a Hölder condition in $\mathfrak{A}$ with exponent α, $0 < \alpha < 1$. Denoting the smallest Hölder coefficient by $H_\alpha(D^m z)$ we may norm these functions as follows

$$\| z \|_m = \max_{\bar{\mathfrak{A}}} | z | + \max_{\bar{\mathfrak{A}}} | Dz | + \cdots + \max_{\bar{\mathfrak{A}}} | D^m z |,$$

$$\| z \|_{m+\alpha} = \| z \|_m + H_\alpha(D^m z),$$

where $D^i z$ represents the i-th order derivative z, $i = 1, \cdots, m$. With the respective norms $\| \ \|_m$ and $\| \ \|_{m+\alpha}$ the classes C_m and $C_{m+\alpha}$ form real Banach spaces. The space C_a is defined for any positive number a. Clearly C_a contains C_b if $a < b$.

We note that *a sphere in* $C_{2+\gamma}$, $0 < \gamma < 1$ *is compact in* $C_{1+\delta}$, $0 < \delta < 1$; that is, the set of z such that $\| z \|_{2+\gamma} \leq H$ (some positive constant) is compact in $C_{1+\delta}$ (with respect to the norm $\| \ \|_{1+\delta}$). More generally, a sphere in C_b is compact in C_a if $a < b$.

We introduce Banach spaces of functions $\phi(s)$ defined on the boundary curve Γ as functions of arc length. Denote by C'_m the class of functions $\phi(s)$ which are m times continuously differentiable, and by $C'_{m+\alpha}$ the subclass of those in C'_m whose m-th derivatives satisfy Hölder conditions with exponent α, $0 < \alpha < 1$. We define the norms of these functions as

$$\| \phi \|'_m = \max_\Gamma | \phi | + \max_\Gamma | \phi' | + \cdots + \max_\Gamma | \phi^{(m)} |,$$

$$\| \phi \|'_{m+\alpha} = \| \phi \|'_m + H'_\alpha(\phi^{(m)}),$$

where $H'_\alpha(\phi^{(m)})$ is the smallest Hölder constant for the m-th derivative $\phi^{(m)}$ of ϕ.

We now make the assumption that the coefficients A, B, C in the differential equation

$$A(x,y,z,p,q)z_{xx} + B(x,y,z,p,q)z_{xy} + C(x,y,z,p,q)z_{yy} = 0 \tag{8.1}$$

are defined for all x, y in the domain $\mathfrak{A}$ and for all values z, p, q. We assume further that for every positive number K, and for all x, y, z, p, q satisfying the conditions

$$(x,y) \text{ in } \mathfrak{A}, \qquad | z |, | p |, | q | \leq K, \tag{8.3}$$

the coefficients A, B, C satisfy

(a) a Hölder condition in the x, y, z, p, q, with coefficients $H(K)$ and exponent $\beta(K)$ depending on the value of K, and

(b) the inequality

$$M(K)(\xi^2 + \eta^2) \geq A\xi^2 + B\xi\eta + C\eta^2 \geq m(K)(\xi^2 + \eta^2) \tag{8.4}$$

for all real ξ, η where $M(K)$ and $m(K)$ are positive constants depending on K.

Condition (*b*) implies that equation (8.1) is elliptic for any values of the arguments inserted into the coefficients.

Let $\phi(s)$ be a given function defined on Γ and assume that $\phi(s) \in C'_{2+\alpha}$ for some α, $0 < \alpha < 1$.

Our existence theorem is contained in

Theorem VI: There exists a solution $z(x,y) \in C_2$ of (8.1) in $\mathfrak{A}$ which is equal to ϕ on the boundary Γ. Furthermore, there exist positive constants $\overline{H}$ and $\gamma < 1$ such that every solution $z \in C_2$ of equation (8.1), which takes on the boundary values ϕ, satisfies the conditions

$$z \in C_{2+\gamma}, \qquad \| z \|_{2+\gamma} \leq \overline{H}. \tag{8.5}$$

Thus there exists a solution z in $C_{2+\gamma}$ of the boundary value problem. (8.5) gives an a priori bound for all solutions (in C_2) of (8.1) taking on the given boundary values.

In No. 7 it is shown that the conditions of the theorem may be weakened. In order that a solution of (8.1) exists it is sufficient that $\phi(s)$ be once differentiable and that the derivative satisfy a Lipschitz condition. The solution will then have continuous second derivatives in $\mathfrak{a}$—not necessarily up to the boundary.

Our definition of the transformation $Z[z]$, described in No. 1, will require an existence theorem for the boundary value problem for a linear elliptic equation. Schauder [19] has derived general existence theorems for linear elliptic equations; we shall make use of one of his theorems (see [19], pages 277–278). He also derived in [19] a priori estimates for solutions of such equations. We state the existence theorem in a form suitable for application to our problem, and include in the statement the a priori estimate for solutions.

Consider a linear elliptic equation

$$a(x,y)Z_{xx} + b(x,y)Z_{xy} + c(x,y)Z_{yy} = 0 \tag{8.6}$$

in the domain $\mathfrak{a}$, where the coefficients a, b, c are in C_μ, $0 < \mu < 1$, and suppose $a\xi^2 + b\xi\eta + c\eta^2 \geq m(\xi^2 + \eta^2)$ for all real ξ,η where m is some positive constant. Let $\phi(s) \in C'_{2+\gamma}$, $0 < \gamma < \mu < 1$, be a given function defined on the boundary of $\mathfrak{a}$.

There exists a unique solution Z in $C_{2+\gamma}$ of (8.6) *taking on the boundary values ϕ. Furthermore, there exists a positive constant k_1 depending only on $\| a \|_\mu$, $\| b \|_\mu$, $\| c \|_\mu$, m and the domain $\mathfrak{a}$, such that the solution taking on the boundary values satisfies the inequality*

$$\| Z \|_{2+\gamma} \leq k_1 \| \phi \|'_{2+\gamma} . \tag{8.7}$$

3. *Bounds on First Derivatives.* Schauder's existence theorem for linear equations will be used to define the transformation $Z[z]$. But, in order to find the appropriate set of functions z which is to be mapped into itself by the transformation $Z[z]$, we proceed to derive estimates for solutions Z of equations of the form (8.2), as outlined in No. 1. We prove first the general

Lemma 5: Let $Z(x,y)$ be a solution of a linear elliptic equation with continuous coefficients

$$a(x,y)Z_{xx} + b(x,y)Z_{xy} + c(x,y)Z_{yy} = 0, \qquad 4ac - b^2 > 0,$$

in $\mathfrak{a}$. Assume that Z is continuous in $\bar{\mathfrak{a}}$ and has continuous second derivatives in $\mathfrak{a}$, and denote the boundary values (on Γ) of Z by $\phi(s)$. Then

$$\| Z \|_1 \leq k \| \phi \|'_2$$

where k is some positive constant depending only on $\mathfrak{a}$.

Proof: Denote the function $Z(x,y)/ \| \phi \|'_2$ by $Z'(x,y)$; we must prove that $\| Z' \|_1 \leq k$, where k is some positive constant. Observe that $Z'(x,y)$ too is a

solution of the differential equation

$$a(x,y)Z'_{xx} + b(x,y)Z'_{xy} + c(x,y)Z'_{yy} = 0$$

and that

$$\| \psi \|'_2 = 1$$

where $\psi(s)$ denotes the boundary values of Z'.

Since the equation for Z' is elliptic, and since only second order derivatives of Z' appear, we may apply the well-known maximum principle[12] for such equations and conclude that

$$\max_{\mathfrak{a}} | Z' | = \max_{\Gamma} | \psi(s) | \leq 1.$$

Furthermore, it follows from the ellipticity of the equation, that the surface in three dimensional space represented by $z = Z'(x,y)$ is a saddle surface, i.e. has non-positive Gauss curvature—for, multiplying the equation by Z'_{xx} we see that

$$aZ'^2_{xx} + bZ'_{xx}Z'_{yy} + cZ'^2_{xy} = c(Z'^2_{xy} - Z'_{xx}Z'_{yy}).$$

Because the differential equation is elliptic, the quadratic on the left side has the same sign as c so that $(Z'_{xx}Z'_{yy} - Z'^2_{xy})$—and hence the Gauss curvature of the surface—is non-positive.

We may now apply a theorem on saddle surfaces, due to T. Radó [16], to obtain the required estimates on the first derivatives of Z'. Radó's theorem concerns a saddle surface represented by an equation $z = Z'(x,y)$ where $Z'(x,y)$ is defined in the closure of a *convex* domain; it was invented for the purpose of obtaining estimates for such surfaces. The theorem states that any plane tangent to an interior point of the surface intersects the boundary curve $x = x(s)$, $y = y(s)$, $z = Z'(x(s), y(s))$ in at least three points; here $x = x(s)$, $y = y(s)$ represents the boundary of the domain. (A particularly simple and elegant proof of the theorem was given by J. von Neumann in [21].)

Thus the problem of estimating the first derivatives of the function $Z'(x,y)$ in $\mathfrak{a}$—i.e. of estimating the slope of any plane tangent to the surface $z = Z'(x,y)$—is reduced to that of finding an estimate of the slope of any plane passing through three points of the boundary curve

$$x = x(s), \qquad y = y(s), \qquad z = \psi(s).$$

One may easily obtain such an estimate for the slope in terms of the maximum of $| \psi(s) | + | \psi''(s) |$ on Γ, i.e. in terms of $\| \psi \|'_2$. The estimate depends on the positiveness of the curvature of Γ. The calculations are not at all difficult and instead of presenting them here we refer to Schauder [18] (pages 626–628); there the argument is clearly presented.

Since $\| \psi \|'_2 = 1$, it follows that the slopes of all such planes are uniformly bounded (the bounds depend on the shape of the domain $\mathfrak{a}$) so that the first

[12]This asserts that z assumes its maximum and minimum on the boundary of $\mathfrak{a}$. For a simple discussion of the maximum principle for elliptic equations see [6].

derivatives of $Z'(x,y)$ are bounded. This fact together with the result obtained from the maximum principle implies

$$\| Z' \|_1 \leq k,$$

where k is a positive constant. Thus Lemma 5 is proved.

If ϕ is the given function on the boundary of $\mathfrak{a}$ we shall denote the constant $k \| \phi \|_2'$ by

$$k \| \phi \|_2' = K.$$

Lemma 5 now implies an a priori estimate for solutions $z' \in C_2$ of the boundary value problem for (8.1),

$$\| z' \|_1 \leq K. \tag{8.8}$$

The constants k and K depend, of course, on the domain $\mathfrak{a}$, but since the domain is fixed throughout, their dependence on $\mathfrak{a}$, and that of other constants, will not be indicated.

4. *A Priori Hölder Inequality for First Derivatives.* In accordance with the outline of No. 1 we use (8.8) to derive new a priori estimates for solutions of (8.1). Denoting the values of the constants $M(K)$ and $m(K)$ of (8.4), with $K = k \| \phi \|_2'$, by M and m, we shall prove

Lemma 6: Let z be a function in C_1 satisfying (8.8)

$$\| z \|_1 \leq K.$$

Assume that a function $Z \in C_2$, taking on the given boundary values ϕ, is a solution of the equation (8.2):

$$\begin{aligned} A(x, y, z, z_x, z_y)Z_{xx} + B(x, y, z, z_x, z_y)Z_{xy} \\ + C(x, y, z, z_x, z_y)Z_{yy} = 0, \end{aligned} \tag{8.9}$$

with the function z and its derivatives inserted in the coefficients A, B, C.

Conclusion: $\| Z \|_1 \leq K$, and there exist positive constants $\overline{K}$ and $\delta < 1$, which depend only on K, m, M and $\| \phi \|_2'$, such that

$$\| Z \|_{1+\delta} \leq \overline{K}. \tag{8.10}$$

Note that the constants $\overline{K}$, δ are independent of the function z.

The lemma yields a new a priori estimate for solutions $z' \in C_2$ of (8.1) taking on the given boundary values — $\| z' \|_{1+\delta} \leq \overline{K}$—if we set $z(x,y)$ and $Z(x,y)$ equal to z'. Thus we have the following a priori estimates for any solution $z' \in C_2$ of (8.1):

$$\| z' \|_1 \leq K = k \| \phi \|_2', \qquad \| z' \|_{1+\delta} \leq \overline{K}. \tag{8.11}$$

Proof of Lemma 6: Since $\| z \|_1 \leq K$, the linear equation for $Z(x,y)$ has coefficients A, B, C satisfying (by condition (*b*) of No. 2) the inequalities

$$M(\xi^2 + \eta^2) \geq A\xi^2 + B\xi\eta + C\eta^2 \geq m(\xi^2 + \eta^2),$$

and is, therefore, elliptic. By Lemma 5 applied to this equation we infer that

$$\| Z \|_1 \leq K,$$

so that, in particular, the first derivatives of Z are bounded by K.

Theorem V may now be applied to the function $Z(x,y)$ as a solution of (8.9). The conditions of the theorem are all satisfied—the constants K, K_1, K_2, λ have the values M, K, $\| \phi \|_2'$, m. We conclude that the first derivatives of $Z(x,y)$ satisfy a Hölder condition in $\mathfrak{A}$ with coefficient C and exponent $\delta < 1$ depending only on M, K, m, $\| \phi \|_2'$. This fact, together with the inequality $\| Z \|_1 \leq K$ imply the conclusion of Lemma 6, with $\overline{K} = K + C$.

5. *The Transformation $Z[z]$ and the Proof of* (8.5). In order to define the transformation $Z[z]$ for a class of functions z it must be possible to solve the linear equation (8.2)

$$a(x,y)Z_{xx} + b(x,y)Z_{xy} + c(x,y)Z_{yy} = 0$$

—for a function Z taking on the given boundary values ϕ—formed by setting $a(x,y) = A(x,y,z,z_x,z_y)$, $b(x,y) = B(x,y,z,z_x,z_y)$, $c(x,y) = C(x,y,z,z_x,z_y)$. According to Schauder's existence theorem for linear elliptic equations, which is stated in No. 2, the equation for Z may be solved, provided that the coefficients a, b, c satisfy Hölder conditions in $\mathfrak{A}$. From the way these coefficients are defined it is clear that this is the case if the first derivatives of the function z satisfy Hölder conditions in $\mathfrak{A}$.

With the aid of this remark we are now in a position to choose the appropriate set of functions z which will be mapped into itself under the transformation $Z[z]$. Lemma 5 implies that the function Z, corresponding to any function z in C_2, satisfies the condition

$$\| Z \|_1 \leq K$$

where $K = k \| \phi \|_2'$. This suggests choosing the set of functions z to satisfy the same condition $\| z \|_1 \leq K$. Furthermore, in order to be able to define the transformation $Z[z]$ for this set of functions z we see from the remark above, that their first derivatives should satisfy Hölder conditions. But what Hölder conditions? Well, Lemma 6 informs us that if z satisfies the condition $\| z \|_1 \leq K$ the corresponding function Z satisfies the condition (8.10), $\| Z \|_{1+\delta} \leq \overline{K}$ suggesting as an additional condition on our set of functions z, the condition $\| z \|_{1+\delta} \leq \overline{K}$.

Let us denote by $S_{1+\delta}$ the set of functions z satisfying the conditions

$$\| z \|_1 \leq K, \qquad \| z \|_{1+\delta} \leq \overline{K}. \tag{8.12}$$

It is clear by the definition of $S_{1+\delta}$ and by Lemmas 5 and 6 that if the transformation $Z[z]$ can be defined for functions z in $S_{1+\delta}$ then it maps the set $S_{1+\delta}$ into itself. We have then to define $Z[z]$ for z in $S_{1+\delta}$.

Denoting the values of the constants $H(K)$ and $\beta(K)$ of condition (a) of No. 2 (with $K = k \| \phi \|_2'$) by H and β, we shall define the transformation $Z[z]$ with the aid of the following

Lemma 7: Let z be in $S_{1+\delta}$, and set $A(x,y,z,z_x,z_y) = a(x,y)$, $B(x,y,z,z_x,z_y) = b(x,y)$, $C(x,y,z,z_x,z_y) = c(x,y)$. There exists a unique solution $Z(x,y) \in C_2$ of the linear elliptic equation

$$a(x, y)Z_{xx} + b(x, y)Z_{xy} + c(x, y)Z_{yy} = 0, \tag{8.13}$$

which takes on the given boundary values $\phi \in C'_{2+\alpha}$. Furthermore, there exist positive constants $\overline{H}$ and γ depending only on K, M, m, $\overline{K}$, δ, H, β and $\| \phi \|'_{2+\alpha}$ (and hence, because of the dependence of $\overline{K}$ and δ, (see No. 4) only on K, M, m, H, β and $\| \phi \|'_{2+\alpha}$), such that

$$Z \in C_{2+\gamma} \quad \text{and} \quad \| Z \|_{2+\gamma} \leq \overline{H}. \tag{8.14}$$

Proof: Since $\| z \|_1 \leq K$, z and its first derivatives p, q satisfy the inequality $| z |, | p |, | q | \leq K$. From condition (a) we see that the functions $A(x,y,z,p,q)$, $B(x,y,z,p,q)$, $C(x,y,z,p,q)$ satisfy Hölder conditions in all five arguments with exponent β and coefficients H. Since, by assumption, the derivatives p, q of z satisfy a Hölder inequality with exponent δ and coefficient $\overline{K}$, for $\| z \|_{1+\delta} \leq \overline{K}$, it follows that the functions $a(x,y)$, $b(x,y)$, $c(x,y)$ satisfy a Hölder condition in $\mathfrak{A}$ with exponent $\beta\delta$ and coefficient depending on K, $\overline{K}$, and H.

Note further that from (8.4) and the inequality $\| z \|_1 \leq K$, the inequalities

$$M(\xi^2 + \eta^2) \geq a\xi^2 + b\xi\eta + c\eta^2 \geq m(\xi^2 + \eta^2) \qquad \text{for real } \xi, \eta$$

follow, with $M = M(K)$, $m = m(K)$ for $K = k \| \phi \|'_2$.

Equation (8.13) is a linear elliptic equation of the type considered in Schauder's existence theorem. The given function $\phi \in C'_{2+\alpha}$ is contained in $C'_{2+\gamma}$ for any positive $\gamma < \alpha$. The constants μ and γ of the theorem may therefore be taken to be, respectively, $\beta\delta$ and any positive number less than $\beta\delta$ and not greater than α. We conclude from the existence theorem that there exists a unique solution Z of (8.13), taking on the given boundary values ϕ. Furthermore

$$Z \in C_{2+\gamma} \quad \text{and} \quad \| Z \|_{2+\gamma} \leq k_1 \| \phi \|'_{2+\gamma}$$

where k_1 is a constant depending on $\| a \|_{\beta\delta}$, $\| b \|_{\beta\delta}$, $\| c \|_{\beta\delta}$ and m. Setting $k_1 \| \phi \|'_{2+\gamma} = \overline{H}$, we see that the proof of Lemma 7 is complete.

It enables us now to define the transformation $Z[z]$ for functions in $S_{1+\delta}$, and in view of Lemma 6 the transformation maps $S_{1+\delta}$ into itself. Furthermore, according to Lemma 7, the image functions Z satisfy the inequality

$$\| Z \|_{2+\gamma} \leq \overline{H}.$$

The last conclusion of Lemma 7 also permits the derivation of new a priori estimates for solutions $z' \in C_2$ of (8.1) taking on the boundary values. It was established that such solutions satisy (8.11)

$$\| z' \|_1 \leq K, \qquad \| z' \|_{1+\delta} \leq \overline{K}.$$

If, in Lemma 7, the function z is set equal to z' then, because the solution Z of (8.13) is unique, Z is also the function z', and hence by Lemma 7 it follows that

$$z' \in C_{2+\gamma} \qquad \text{and} \qquad \| z' \|_{2+\gamma} \leq \overline{H}.$$

We have, therefore, derived the a priori estimate (8.5) of Theorem VI.

In the estimate just obtained for z' the Hölder exponent γ is less than $\beta\delta$. Using the estimate in the manner described in No. 1 we may derive an a priori estimate for $\| z' \|_{2+\gamma'}$ with γ' any positive number less than β and $\leq \alpha$. Consider equation (8.1) satisfied by z', and set $A(x, y, z', z'_x, z'_y) = a(x, y)$, $B(x, y, z', z'_x, z'_y) = b(x, y)$, $C(x, y, z', z'_x, z'_y) = c(x, y)$, so that the equation takes the form

$$az'_{xx} + bz'_{xy} + cz'_{yy} = 0.$$

Since the second derivatives of z' are bounded by $\overline{H}$, so that the first derivatives satisfy Lipschitz conditions with the coefficient $\overline{H}$, it follows that the functions a, b, c satisfy Hölder conditions with exponent β. Applying the Schauder theorem of No. 2 to the equation we conclude that the second derivatives of z' satisfy Hölder conditions with any exponent less than β and not greater than α. Thus if $\beta > \alpha$ we may conclude that the second derivatives of z' satisfy a Hölder condition with exponent α.

In particular if the coefficients A, B, C satisfy Lipschitz conditions in all the arguments x, y, z, p, q, i.e., if we assume $\beta(K) = 1$, we infer that the solution $z'(x,y)$ is in $C_{2+\alpha}$ and satisfies the a priori estimate

$$\| z' \|_{2+\alpha} \leq H',$$

where H' is a constant depending only on K, M, m, H and $\| \phi \|'_{2+\alpha}$.

6. *Proof of Theorem VI.* The transformation $Z[z]$ has been defined for functions z in the set $S_{1+\delta}$, i.e. functions z satisfying the inequalities

$$\| z \|_1 \leq K, \qquad \| z \|_{1+\delta} \leq \overline{K},$$

and maps this set into itself. Furthermore, the transformation maps the functions z into functions Z which lie in $C_{2+\gamma}$ and satisfy (8.14)

$$\| Z \|_{2+\gamma} \leq \overline{H}.$$

Since the set $S_{1+\delta}$ is convex we assert that the conditions of Schauder's fixed point theorem of No. 1 are satisfied by the transformation in the Banach space $C_{1+\delta}$. For, (*1*) $Z[z]$ maps the convex set $S_{1+\delta}$ into itself, (*2*) the mapping is completely continuous (that it is continuous is just as easily shown)—this follows from the fact that the image points lie in a sphere $\| Z \|_{2+\gamma} \leq \overline{H}$ in $C_{2+\gamma}$, and are therefore compact in $C_{1+\delta}$.

Therefore the transformation $Z[z]$ has a fixed point $z'(x,y)$. Since the image points of the transformation lie in $C_{2+\gamma}$ and satisfy

$$\| Z \|_{2+\gamma} \leq \overline{H}$$

it follows that the same is true of the fixed point $z'(x,y)$. Therefore $z'(x,y)$ solves the differential equation (8.1) and the proof of Theorem VI is complete.

7. *Weakening of Assumptions on Boundary Values.* It has already been pointed out (see p. 139) that we may show the existence of a solution of (8.1) taking on given boundary values $\phi(s)$, and having continuous second derivatives in $\mathfrak{A}$ but not necessarily in $\bar{\mathfrak{A}}$, if we assume merely that the function $\phi(s)$ is once differentiable and that the first derivative $\phi'(s)$ satisfies a Lipschitz condition. (That is, we assume that there exists a constant L such that $|\phi'(s_1) - \phi'(s_2)| \leq L|s_1 - s_2|$ for all s_1, s_2.) For this purpose we need the following theorem, due to Schauder ([19], Theorem 1, p. 265) giving a priori estimates of solutions of a linear elliptic equation in closed subdomains of the domain. Again we state this in a form best suited for application here.

Let $Z(x,y)$, a bounded function having continuous first and second derivatives in a bounded domain $\mathfrak{A}$, satisfy a linear elliptic equation in $\mathfrak{A}$

$$a(x,y)Z_{xx} + b(x,y)Z_{xy} + c(x,y)Z_{yy} = 0,$$

where the coefficients a, b, c are in C_μ, $0 < \mu < 1$. Suppose that $a\xi^2 + b\xi\eta + c\eta^2 \geq m(\xi^2 + \eta^2)$ for all real ξ, η, where m is some positive constant. *For every positive number $\gamma < \mu$ and every closed subdomain $\mathfrak{B}$ of $\mathfrak{A}$*

$$Z(x,y) \in C_{2+\gamma} \qquad \text{in } \mathfrak{B},$$

and

$$\| Z \|^{\mathfrak{B}}_{2+\gamma} \leq k_1 \max_{\mathfrak{A}} |Z|.$$

Here $\| \quad \|^{\mathfrak{B}}_{2+\gamma}$ denotes the norm of the function considered as being defined only in the closed domain $\mathfrak{B}$; k_1 is a constant depending only on γ, μ, $\| a \|_\mu$, $\| b \|_\mu$, $\| c \|_\mu$, m and the distance d from $\mathfrak{B}$ to the boundary of $\mathfrak{A}$.

That Z is in $C_{2+\gamma}$ follows from Theorem I, of [5] (page 208).

Suppose now that we wish to find a solution of (8.1) assuming the given boundary values $\phi(s)$. Set $\max\{\| \phi \|'_1, L\} = \kappa$, where L is the Lipschitz constant of ϕ'. Approximate $\phi(s)$, in the sense of the norm $\| \quad \|'_1$, by a sequence of functions $\phi_n(s)$ having continuous derivatives up to the third order and such that $\| \phi_n \|'_2 \leq 2\kappa$, $n = 1, 2, 3, \cdots$. We know that for each of these functions we may solve equation (8.1) for a function $z_n(x,y)$ which equals $\phi_n(s)$ on the boundary—this follows from Theorem VI. By the a priori bounds (8.11) established for solutions of (8.1) we know that $\| z_n \|_1 \leq k \| \phi_n \|'_2 = 2k\kappa$ and that $\| z_n \|_{1+\delta'} \leq \overline{K'}$ where $\overline{K'}$ and δ' are constants, $0 < \delta' < 1$, which depend on κ and are independent of n. It follows, as in the proof of Lemma 7, that the functions $A(x, y, z_n, p_n, q_n)$, $B(x, y, z_n, p_n, q_n)$, $C(x, y, z_n, p_n, q_n)$ as functions of x and y satisfy a uniform Hölder condition (independent of n) in $\mathfrak{A}$. We may therefore apply the just stated theorem by Schauder to the equation satisfied by z_n, and conclude that there exists a positive constant $\gamma < 1$, such that for any closed subdomain $\mathfrak{B}$ of $\mathfrak{A}$ the inequality

$$\| z_n \|^{\mathfrak{B}}_{2+\gamma} \leq k_1 \max_{\mathfrak{A}} |z_n|, \qquad n = 1, 2, \cdots$$

holds, where k_1 is a constant independent of n. Since $\| z_n \|_1 \leq 2k_K$ we have

$$\| z_n \|^{\mathfrak{A}}_{2+\gamma} \leq 2k_1 k_K.$$

This last inequality implies that the functions z_n and their first and second derivatives are uniformly bounded and equicontinuous (since the second derivatives satisfy a uniform Hölder condition) in every closed subdomain of $\mathfrak{A}$. It follows, in the usual way, that we may select a subsequence of the z_n which will converge (together with first and second derivatives) to a function $z(x,y)$ (and its first and second derivatives) in $\mathfrak{A}$. In every closed subdomain of $\mathfrak{A}$ the convergence will be uniform and the second derivatives of $z(x,y)$ satisfy a Hölder condition.

Clearly $z(x,y)$ is a solution of the differential equation (8.1). Since it satisfies the inequality

$$\| z \|_1 \leq 2k_K,$$

it is continuous in the closure of $\mathfrak{A}$ and because $\| \phi - \phi_n \|'_1 \to 0$, assumes the value $\phi(s)$ on the boundary.

8. *Solution of Boundary Value Problem Using Theorem IV.* Our proof of Theorem VI made use of Theorem V only in the proof of Lemma 6 showing that the first derivatives of the solution Z of (8.9) satisfy a Hölder condition in $\mathfrak{A}$. It is of interest that one may demonstrate the existence of a solution of (8.1), which takes on the boundary values ϕ, using—instead of Theorem V—Theorem IV. The solution so obtained is continuous in $\bar{\mathfrak{A}}$ and has continuous second derivatives in $\mathfrak{A}$, but not necessarily in $\bar{\mathfrak{A}}$.

In order to carry out this existence proof one constructs again an appropriate class of functions z for which the transformation $Z[z]$ may be defined, and which is mapped into itself under this transformation—thus enabling the use of the Schauder fixed point theorem. Our class of functions z in the proof already given (defined in No. 5 as $S_{1+\delta}$) was determined by means of Lemmas 5 and 6 which made use of Theorem V. Suppose now, in the discussion of Lemma 6, we use Theorem IV instead of V, what is the corresponding class of functions z so obtained? It is the nature of this class of functions, and of the corresponding Banach space, that is the interesting feature here.

To determine this class consider again the differential equation (8.9) for Z (of Lemma 6) with z satisfying condition (8.8). Going through the proof of Lemma 6, but applying Theorem IV instead of Theorem V, we conclude that the first derivatives of Z satisfy a Hölder inequality in *any closed subdomain* of $\mathfrak{A}$ (not necessarily in all of $\mathfrak{A}$).

Thus if we denote by $\mathfrak{A}_n$ the domain consisting of those points of $\mathfrak{A}$ whose distance from the boundary of $\mathfrak{A}$ is greater than $1/n$, $n = 1, 2, \cdots$, we may conclude, in particular, that there exist positive constants $\delta_n < 1$ and $\bar{K}_n$, depending only on m, K, M, n and $\| \phi \|'_2$, such that the inequalities

$$\| Z \|^{\mathfrak{A}_n}_{1+\delta_n} \leq \bar{K}_n \qquad n = 1, 2, \cdots, \tag{8.15}$$

hold. Here the left side represents the norm of Z considered as a function defined in the domain $\mathfrak{a}_n$.

Inequalities (8.15) replace the inequality (8.10) which was obtained with the aid of Theorem V. Lemma 5 and inequalities (8.15) now suggest (as in No. 5) that the appropriate set of functions for which the transformation $Z[z]$ is to be defined is the set of functions, which we denote by $\tilde{S}$, satisfying the conditions

$$\| z \|_1 \leq K, \qquad \| z \|^{\mathfrak{a}_n}_{1+\delta_n} \leq \overline{K}_n, \qquad n = 1, 2, \cdots.$$

By Lemma 5 and the inequalities (8.15) it is clear that the transformation $Z[z]$, if it can be defined on $\tilde{S}$, maps $\tilde{S}$ into itself.

In order to complete the existence proof, using Schauder's fixed point theorem we must show the following:

(a) The transformation $Z[z]$ may be defined for functions z in $\tilde{S}$. That is, for any function z of $\tilde{S}$ there exists a unique solution Z of the equation

$$A(x,y,z,z_x,z_y)Z_{xx} + B(x,y,z,z_x,z_y)Z_{xy} + C(x,y,z,z_x,z_y)Z_{yy} = 0, \tag{8.16}$$

having continuous second derivatives in $\mathfrak{a}$, and taking on the given boundary values ϕ.

(b) The set $\tilde{S}$ is a convex set lying in a Branach space $\tilde{C}$.

(c) The transformation $Z[z]$ is completely continuous in $\tilde{C}$. (It must also be shown that $Z[z]$ is continuous in $\tilde{C}$; this is easily done, and we omit it here.)

Assuming (a), we show first (b) and (c).

Proof of (*b*): The definition of the appropriate Banach space $\tilde{C}$ is suggested by inequalities (8.15). Consider any function $z(x,y)$ having continuous first derivatives in $\mathfrak{a}$ which satisfy Hölder inequalities in $\mathfrak{a}_n$ with exponent δ_n for all $n = 1, 2, \cdots$. Define a new norm for $z(x,y)$ by

$$\widetilde{\| z \|} = \underset{n}{\text{l.u.b.}} \frac{1}{K_n} \| z \|^{\mathfrak{a}_n}_{1+\delta_n}.$$

The Banach space $\tilde{C}$ is now defined as the set of functions z having finite norm $\widetilde{\| z \|}$.

The set $\tilde{S}$ may then be characterized as the set of functions z satisfying the inequalities

$$\| z \|_1 \leq K, \qquad \widetilde{\| z \|} \leq 1,$$

and is clearly a convex set in $\tilde{C}$.

Proof of (*c*): In order to demonstrate the complete continuity of the transformation $Z[z]$ in $\tilde{C}$ we establish estimates for the solution Z of (8.16) with z in $\tilde{S}$. This is done with the aid of Schauder's theorem on a priori estimates of solutions of elliptic equations quoted in No. 7.

Let $Z(x,y)$ be the solution of (8.16) taking on the given boundary values

ϕ, with z in $\tilde{S}$—we are assuming that (a) is verified. Consider the equation in the domain $\mathfrak{A}_{n+1}$ (for any n). In that domain the first derivatives of z are bounded by K and satisfy a Hölder condition with exponent δ_{n+1} and coefficient $\overline{K}_{n+1}$ (since z is in $\tilde{S}$). It follows, as in the proof of Lemma 7, that the coefficients $a(x,y) = A(x,y,z,z_x,z_y), \cdots$ of the equation (8.16), regarded as known functions of x and y, satisfy in $\mathfrak{A}_{n+1}$ a Hölder condition with exponent and coefficient depending only on β, δ_{n+1}, K, $\overline{K}_{n+1}$ and H; furthermore, they satisfy the following inequality

$$M(\xi^2 + \eta^2) \geq a\xi^2 + b\xi\eta + c\eta^2 \geq m(\xi^2 + \eta^2) \qquad \text{for all real } \xi,\eta.$$

(The definitions of the constants M and m are given in No. 4 while the definitions of β and H precede the statement of Lemma 7 in No. 5.) We may therefore apply Schauder's theorem of No. 7 to the equation (8.16) in the domain $\mathfrak{A}_{n+1}$ and conclude that in the closure of $\mathfrak{A}_n$, which is a closed subdomain of $\mathfrak{A}_{n+1}$, the solution Z has continuous second derivatives satisfying Hölder conditions with exponent γ_n and that

$$\| Z \|_{2+\gamma_n}^{\mathfrak{A}_n} \leq k_n \max | Z | \leq k_n \max | \phi |$$

where γ_n and k_n depend only on the constants β, δ_{n+1}, K, $\overline{K}_{n+1}$, H, M, m and n.

The complete continuity of the transformation $Z[z]$ now follows from the fact that the set of functions Z satisfying the inequalities

$$\| Z \|_{2+\gamma_n}^{\mathfrak{A}_n} \leq k_n \max | \phi | \qquad n = 1, 2, \cdots$$

is compact in $\tilde{C}$.

Thus, in order to solve the boundary value problem, by proving the existence of a fixed point of the transformation $Z[z]$, we have only to verify (a). Note that the solution $z(x,y)$, so obtained as a fixed point, satisfies the inequalities

$$\| z \|_{2+\gamma_n}^{\mathfrak{A}_n} \leq k_n \max | \phi |. \tag{8.17}$$

Proof of (*a*): Consider equation (8.16) with z some function in $\tilde{S}$. It has the form

$$a(x,y)Z_{xx} + b(x,y)Z_{xy} + c(x,y)Z_{yy} = 0, \tag{8.18}$$

and is to be solved for the function Z taking on the given boundary values ϕ. As remarked above in the proof of (b) the coefficients a, b, c satisfy in each domain $\mathfrak{A}_n$ a Hölder condition[13]—with exponent and coefficient which we now denote by α_n and K_n. The existence of a solution Z will be proved by approximating the coefficients a, b, c by functions a_n, b_n, c_n, $n = 1, 2, \cdots$, which agree with a, b, c in $\mathfrak{A}_n$, and which satisfy Hölder conditions in the whole domain $\mathfrak{A}$. The analogous linear differential equations with coefficients a_n, b_n, c_n will admit solutions taking on the given boundary values; a subsequence of these solutions will converge to the solution of (8.16).

[13]The existence proof given here applies to all equations of the form (8.18) (and to a wider class) with coefficients satisfying Hölder conditions in every closed subdomain of the full domain.

The approximation of the coefficients a, b, c by functions a_n, b_n, c_n having the required properties is particularly simple in our case: For n sufficiently large, say $n >$ some number N, the boundry of $\mathfrak{A}_n$ is a convex curve having positive curvature. To every point (x,y) in $\mathfrak{A}$ outside of $\mathfrak{A}_n$ there is a unique nearest point (x',y') in $\bar{\mathfrak{A}}_n$. We define the approximating functions a_n, b_n, c_n for $n > N$ by

$$a_n, b_n, c_n = \begin{cases} a(x,y),\ b(x,y),\ c(x,y) & \text{for } (x,y) \text{ in } \mathfrak{A}_n, \\ a(x',y'),\ b(x',y'),\ c(x',y') & \text{for } (x,y) \text{ not in } \mathfrak{A}_n, \end{cases}$$

$n = N + 1, \cdots$. The functions a_n, b_n, c_n satisfy in $\mathfrak{A}$ a Hölder condition (which varies with n). Therefore, by Schauder's existence theorem for linear elliptic equations we may solve the equation

$$a_n Z_{nxx} + b_n Z_{nxy} + c_n Z_{nyy} = 0$$

for a function $Z_n(x,y)$ which takes on the given boundary values ϕ $(n = 1, 2, \cdots)$.

Consider now the functions Z_{j+1}, Z_{j+2}, $\cdots$ in the domain $\mathfrak{A}_{j+1}$ for some fixed j. In $\mathfrak{A}_{j+1}$ the coefficients a_i, b_i, c_i for all $i \geq j + 1$, are equal to a, b, c and hence satisfy a Hölder condition with exponent α_{j+1} and coefficients K_{j+1}. Using now Schauder's theorem on a priori estimates in closed subdomains, we conclude that there exist positive constants k_j, and $\gamma_j < 1$, such that

$$\| Z_i \|^{\mathfrak{A}_j}_{2+\gamma_j} \leq k_j \max | Z_i | \leq k_j \max_{\Gamma} | \phi |;$$

for, the closure of $\mathfrak{A}_j$ is a closed subdomain of $\mathfrak{A}_{j+1}$. Thus the functions Z_{j+1}, Z_{j+2}, $\cdots$ and their first and second derivatives are uniformly bounded and equicontinuous in $\mathfrak{A}_j$, and we may therefore select a subsequence which converges (together with first and second derivatives) to a function $Z(x,y)$ (and its derivatives) in $\mathfrak{A}_j$. By letting j progress through $N + 1, N + 2, \cdots$ we may, by the usual diagonalization process, find a subsequence of the Z_n which will converge (together with first and second derivatives) to a function $Z(x,y)$ (and its corresponding derivatives) in all of $\mathfrak{A}$. Clearly $Z(x,y)$ is the required solution. That $Z(x,y)$ assumes the given boundary values and is continuous in the closure of $\mathfrak{A}$ follows from the fact that $Z(x,y)$ satisfies the inequality—as do the functions $Z_n(x,y)$—

$$\| Z \|_1 \leq k \| \phi \|_2'$$

which is a consequence of Lemma 5. Thus (a) is verified, and the solution of the boundary value problem for equation (8.1) is complete.

It follows from (8.17) that the second derivatives of the solution $z(x,y)$ of (8.1) so obtained satisfy Hölder conditions in every closed subdomain of $\mathfrak{A}$.

9. *Estimates for Higher Derivatives of Solutions of Nonlinear Elliptic Equations*

1. It is easily seen that with the aid of the theory of linear elliptic equations developed in [19] one may obtain estimates for derivatives of all orders of a

solution of the nonlinear elliptic equation (1.1), once bounds for the solutions and its derivatives up to third order are known. In this section we show how to obtain estimates for derivatives of order greater than two of a solution of (1.1), knowing bounds for its first and second derivatives. The estimates we obtain are of two kinds: estimates of derivatives in closed subdomains, and, under additional assumptions concerning smoothness of the boundary and of the boundary values, estimates for derivatives in the whole domain.

The estimates of the first kind follow immediately from Theorem I and from a priori estimates for solutions of *linear* elliptic equations, as given by Schauder in [19]. Those of the second kind are derived using a sharp form of Theorem I (which is proved below) and, again, Schauder's estimates in [19] for *linear* elliptic equations.

Estimates of the second kind for the general nonlinear elliptic equation were first given by S. Bernstein [2], and were re-established by Schauder using Bernstein's method of 'auxiliary functions' ([18] section 6). Schauder's estimates are not quite as strong as those given here. He requires, for instance, more differentiability of the differential equations (see also footnote 15 on page 154). One main feature of our procedure for obtaining the estimates is that given Theorem I (or its sharp form) we need only use statements concerning *linear* elliptic equations. Schauder's use of the auxiliary functions (which is somewhat difficult to follow) involves more essentially the nonlinear character of the equation.

The estimates of the first kind for the general nonlinear elliptic equation, which we present here are new. H. Lewy [11] derived such estimates for nonlinear elliptic equations of the Monge-Ampère type which are analytic. In addition, however, he succeeded in obtaining a priori estimates for derivatives of *second* order of solutions of a class of such equations ([11], II).

In connection with the problem of finding a priori estimates, depending on the boundary values, of derivatives of solutions of general nonlinear elliptic equations, mention should be made of the work of J. Leray [7], [8].

In [7] he obtained a priori bounds for derivatives of *second* order of solutions of a class of nonlinear elliptic equations (the class includes quasilinear equations) in terms of bounds for derivatives of first order. In [8] he discussed, still further, equations for which a priori bounds for derivatives of first order of solutions may be derived, and described classes of equations for which such bounds do not exist. In particular he furnished criteria for the existence or non-existence of a priori bounds for derivatives (of first and second order) of solutions of equations of Monge-Ampère type.

2. *Estimates of the first kind.* We shall consider a solution $z(x,y)$ of an elliptic equation

$$F(x,y,z,p,q,r,s,t) = 0 \tag{9.1}$$

in a bounded domain D, and assume that the conditions of Theorem I are satisfied, i.e.

(i) The first partial derivatives of F (with the values of $z(x,y)$ and its derivatives inserted into the arguments) are bounded in absolute value by a constant K.

(ii) $z(x,y)$ has continuous first and second derivatives bounded by K_1.

(iii) For any real (ξ,η) the inequality

$$F_r\xi^2 + F_s\xi\eta + F_t\eta^2 \geq \lambda(\xi^2 + \eta^2)$$

holds for all (x,y) in D, with λ a positive constant.

Furthermore, assume

(iv) F has continuous derivatives up to order m with respect to all variables and its derivatives of m-th order satisfy Hölder conditions with respect to all variables. Let the constant K (of (i)) be a bound for the derivatives of F up to order m and for the Hölder coefficient of the derivatives of m-th order—when the values of $z(x,y)$ and its derivatives are inserted into the arguments. Let β denote the exponent of this Hölder condition.

It follows from Theorem III that the solution $z(x,y)$ possesses partial derivatives up to order $m + 2$ in D and that its derivatives of order $m + 2$ satisfy a Hölder condition in any closed subdomain of D.

The estimates of the first kind for the solution $z(x,y)$ are contained in

Theorem VII: In any closed connected subdomain $\mathfrak{B}$ of $\mathfrak{D}$ the derivatives of $z(x,y)$ up to order $m + 2$ are bounded in absolute value by a constant which, together with the constants of the Hölder inequality in $\mathfrak{B}$ for the derivatives of order $m + 2$, depends only on the constants K, K_1, λ, β, the distance d from $\mathfrak{B}$ to the boundary of $\mathfrak{D}$, and the diameter D of $\mathfrak{D}$[14].

Proof: To prove this theorem we differentiate equation (9.1) and apply known theorems on linear equations. We restrict ourselves to the proof for the case $m = 1$ where estimates for third derivatives must be found. Estimates for derivatives of higher order are obtained by further differentiation of the equation and application of the same argument.

Differentiate equation (9.1) with respect to x and consider the resulting equation as a linear equation in $p = z_x$ with known coefficients

$$F_r p_{xx} + F_s p_{xy} + F_t p_{yy} + F_p p_x + F_q p_y + F_z p + F_x = 0. \tag{9.2}$$

We claim that the arguments $x, y, z \cdots, z_{yy}$ occurring in the coefficients, considered as functions of (x,y), satisfy in any connected closed subdomain $\mathfrak{B}'$ of $\mathfrak{D}$ a Hölder inequality with constants depending only on the constants of (i)-(iii), the distance d' from $\mathfrak{B}'$ to the boundary of $\mathfrak{D}$, and D. For the arguments z_{xx}, z_{xy}, z_{yy} the Hölder inequality is a consequence of Theorem I. For the arguments z_x, z_y it is a consequence of condition (ii). Finally, it is easily seen, using the connectivity of $\mathfrak{B}'$, that the argument $z(x,y)$ satisfies a Hölder inequality in $\mathfrak{B}'$ for which the constants depend only on K_1 (of (ii)), d' and D.

[14] If a bound for the function z is known the assumptions of the boundedness of the domain $\mathfrak{D}$ and the connectedness of $\mathfrak{B}$ are unnecessary.

It follows now, from the Hölder continuity (in all arguments) of the first derivatives of F, that the coefficients in equation (9.2), considered as known functions of (x,y) satisfy a Hölder condition in $\mathfrak{B}'$ with constants depending only on the constants of (i)-(iv), on d', and D.

Let $\mathfrak{B}$ be a given connected closed subdomain of $\mathfrak{D}$. Introduce another closed subdomain $\mathfrak{B}'$ of $\mathfrak{D}$ consisting of the connected component containing $\mathfrak{B}$ of the set of all points whose distance from the boundary of $\mathfrak{D}$ is not less than $d/2$: clearly $d' = d/2$ and $\mathfrak{B}$ is a closed subdomain of $\mathfrak{B}'$. To equation (9.2), considered in $\mathfrak{B}'$, we apply a theorem on a priori bounds for solutions of linear elliptic equations due to Schauder (Theorem I, p. 265 in [19])—of which the theorem stated on page 145 is a special case—and conclude that in $\mathfrak{B}$ the second derivatives of p, i.e. z_{xx}, z_{xy} and z_{yy} are bounded by a constant which, together with the constants of the Hölder inequalities for these second derivatives in $\mathfrak{B}$, depends only on the constants of (i)-(iv), on d and on D.

Similarly, differentiating (9.1) with respect to y we obtain analogous estimates for z_{yyy}, thus proving Theorem VII.

3. *Estimates of the second kind. A sharp form of Theorem I.* Estimates of the second kind for solutions of (9.1) will be derived for solutions $z(x,y)$ satisfying conditions (i)-(iv) above and the additional conditions:

(v) (a) The domain $\mathfrak{D}$ is of type L_{m+2} (see §2, 1), and the function $z(x,y)$ and its derivatives up to order $m + 2$ are continuous in the closure $\bar{\mathfrak{D}}$ of $\mathfrak{D}$. In addition the boundary values of $z(x,y)$, regarded as functions of arc length, have continuous derivatives up to order $m + 2$ which are bounded by a constant K_2.

(b) Furthermore, the derivative of order $m + 2$ of the boundary values satisfies a Hölder condition with coefficient K_2 and exponent γ.

These estimates for the solution $z(x,y)$ are contained in

Theorem VIII: The derivatives of $z(x,y)$ up to order $m + 2$ are bounded in absolute value in $\bar{\mathfrak{D}}$, and those of order $m + 2$ satisfy in $\bar{\mathfrak{D}}$ a Hölder condition, the constants of which, together with the bound for all the derivatives, depend only on the constants of (i)-(v), and on the domain $\mathfrak{D}$.

In order to prove Theorem VIII we shall make use of a strong form of Theorem I:

Theorem IX: Let $z(x,y)$ be a solution of (9.1) and assume that conditions (i)-(iv) and (va) are satisfied for $m = 1$. Then the derivatives of second order satisfy in $\bar{\mathfrak{D}}$ a Hölder condition with constants depending only on the constants of (i)-(iii), (va), and on the domain $\mathfrak{D}$.

Proof: The proof being similar to that of Theorem V which was carried out in detail, is presented merely in outline. It would, of course, be very convenient if we could apply Theorem V directly to equation (9.2), and derive a Hölder inequality in $\mathfrak{D}$ for the first derivatives of p, i.e. for r and s. Indeed $p(x,y)$ and equation (9.2) satisfy all the conditions of Theorem V—except that,

on the boundary we are given estimates for the third derivative of the boundary values of z, not for the second derivative of the boundary values of p.

The procedure to be followed is, as in §6, to establish estimates of the form (6.4) for the functions r, s and t, that is, to find positive constants d, M, $\alpha < 1$, depending only on the constants of (i)–(iii), (va), and on the domain $\mathfrak{D}$, such that the inequality

$$\iint_{C_d} \rho^{-\alpha}(r_x^2 + r_y^2 + s_x^2 + s_y^2 + t_x^2 + t_y^2)\, dx\, dy \leq M \tag{9.3}$$

holds, where C_d represents the intersection of $\mathfrak{D}$ with any circle having centre in $\mathfrak{D}$ and radius d, and ρ is the distance of point of integration from the centre. The desired Hölder inequality for r, s and t then follows from Lemma 1′.

The estimates of the form (9.3) will be derived first for the functions r and s, which satisfy an inequality

$$r_x^2 + r_y^2 + s_x^2 + s_y^2 \leq k(r_y s_x - r_x s_y) + k_1\,, \tag{9.4}$$

where k and k_1 are non-negative constants depending only on the constants of (i)–(iii). This inequality is a consequence of the Remark at the beginning of §4 applied to (9.2). The corresponding estimate, of the form (9.3) for t follows from the inequality

$$t_x^2 + t_y^2 \leq \overline{K}(r_x^2 + r_y^2 + s_x^2 + s_y^2 + 1), \tag{9.5}$$

where $\overline{K}$ is a constant depending on the constants of (i)–(iii). Inequality (9.5) is derived immediately from the quasilinear equation obtained by differentiating equation (9.1) with respect to y.

Applying Lemma 3 to (9.4) it is seen that the functions r and s satisfy estimates of the form (9.3) for sufficiently small circles C_d lying entirely inside $\mathfrak{D}$ (and bounded away from the boundary of $\mathfrak{D}$). We shall have to derive such estimates for the C_d which may approach, and even intersect, the boundary of $\mathfrak{D}$.

To this end, as in §6, 3, we introduce local transformations of variables (from (x,y) to (ξ,η), of the type (6.5)), in the neighborhood of a boundary curve, mapping the boundary curve, at least locally, into a straight segment (on $\eta =$ constant). The function $z(x,y)$ as a function $z'(\xi,\eta)$ of the new variables satisfies a transformed differential equation, which we may refer to as (9.1)′. On differentiating (9.1)′ with respect to ξ (i.e. in a direction parallel to the straight boundary segment) we find, as in §6, 3, that the second derivatives $z'_{\xi\xi} = r'$, $z_{\xi\eta} = s'$ satisfy an inequality similar to (9.4), which we may refer to as (9.4)′, with constants k' and k'_1 depending only on the constants of (i)–(iii) and the domain $\mathfrak{D}$ (here we use the fact that $\mathfrak{D}$ is of type L_2). In addition, since a bound for the third derivative of the boundary values of z is known (from (va)) we note that on the straight segment of the transformed boundary

$$|\, r'_\xi \,| \leq K'_2\,,$$

where the constant K_2' depends only on the constants of (i)–(iii), (va), and the domain $\mathfrak{D}$. In virtue of (9.4)′ and this last inequality we may apply Lemma 3′ and conclude that estimates of the form (9.3) hold for the functions r' and s' in circles C_d' which may intersect the straight boundary segment. Differentiating (9.1)′ with respect to η we may derive the analogue (9.5)′ of (9.5), and hence conclude that t' also satisfies estimates of the form (9.3) for the circles C_d'.

Reintroducing the variables x and y we may duplicate the rest of the argument given in §6, 3 and obtain the numbers d, M, $\alpha < 1$ for which (9.3) holds, and thus complete the proof of Theorem IX.

Remark: It is possible, following the remark at the end of §7, to establish the Hölder conditions for r, s and t, in $\mathfrak{D}$, in terms of the constants of (i)–(iii), a constant K_3, and the domain $\mathfrak{D}$; here K_3 is a bound on the integrals of the squares of the third derivatives of the boundary values of z (with respect to arc length) along the boundary curves.

4. *Proof of Theorem VIII.* As in the proof of Theorem VII we describe here only the proof for the case $m = 1$, where we must find estimates for third derivatives of z.[15] Again we differentiate equation (9.1) and apply Theorem IX and theorems on linear equations. The estimates for derivatives of higher order ($m > 1$) may then be obtained by further differentiation and repetition of the same argument.

The desired estimates for closed subdomains are given by Theorem VII; in order to derive the estimates for points near the boundary we introduce, as in No. 3, local transformations (from (x,y) to (ξ,η), of the type (6.5)), in the neighborhood of a boundary curve, mapping the boundary curve, at least locally, into a straight segment Γ (on η = constant). The transformed function $z'(\xi,\eta)$ satisfies the transformed equation, which we again refer to as (9.1)′, and which, on differentiation with respect to ξ yields an equation (9.2)′, analogous to (9.2), which we may consider as a 'linear' equation with known coefficients in the function $z_\xi' = p'$. This equation holds in a domain Γ' having Γ as part of its boundary.

As in the proof of Theorem VII we may conclude, using Theorem IX instead of Theorem I, that the coefficients of equation (9.2)′ satisfy, as functions of ξ and η in this domain Γ', a Hölder condition with constants depending on the constants of (i)–(iv) and on $\mathfrak{D}$. In addition the solution p' of the 'linear' equation (9.2)′ satisfies on Γ the inequality

$$|p_{\xi\xi}'| \leq K_2',$$

in virtue of (va), and $p_{\xi\xi}'$ satisfies on Γ a Hölder condition (in virtue of (vb)), with constants which, together with K_2', depend on the constants of (i)–(v) and on $\mathfrak{D}$.

[15] As Theorem VII asserts, the estimates so obtained depend on estimates for the derivatives of F of *first* order only. The corresponding estimates obtained by Bernstein and Schauder (see No. 1) depend also on estimates for derivatives of F of second order.

In his study of linear elliptic equations with Hölder continuous coefficients Schauder showed how to obtain a priori estimates for the second derivatives (and their Hölder continuity) of a solution of the equation in a subdomain $\mathfrak{A}$ (of the original domain Γ' where the differential equation holds) which may abut a boundary segment Γ of Γ', provided that (a) $\mathfrak{A}$ remains bounded away from any other boundary points of Γ', (*b*) on Γ estimates for the derivatives of the boundary values up to second order and their Hölder continuity is known. (See [13], Chapter 3, in particular Theorem 2.) This is exactly our situation, and using Schauder's results, we may obtain estimates for the second derivatives of p', i.e. for $z'_{\xi\xi}$, $z'_{\xi\eta}$, $z'_{\eta\eta}$, and their Hölder continuity in a subdomain of Γ' lying near Γ. From the equation obtained by differentiating (9.1)′ with respect to η, we see that we can express $z'_{\eta\eta\eta}$ in terms of these other third derivatives of z'. Hence similar estimates hold for $z'_{\eta\eta\eta}$.

On re-introducing the original coordinates (x,y), and noting that this procedure may be carried out in a neighborhood of every boundary point of $\mathfrak{D}$, we obtain estimates for the third derivatives of z and their Hölder continuity in a domain consisting of points lying in some neighborhood of the boundary of $\mathfrak{D}$. (The fact that $\mathfrak{D}$ is of type L_2 is used here. The details of this procedure are related to those carried out in §6, 3 and are not presented.)

The remainder of $\overline{\mathfrak{D}}$ is a closed subdomain of $\mathfrak{D}$ for which such estimates have been established in Theorem VII. Combining these estimates by a simple argument we may derive the required estimates in the whole domain $\mathfrak{D}$, and thus complete the proof of Theorem VIII.

Bibliography

[1] Bernstein, S., *Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre*, Mathematische Annalen, Volume 59, 1904, pp. 20-76.

[2] Bernstein, S., *Sur la généralisation du problème de Dirichlet*, Mathematische Annalen, Volume 69, 1910, pp. 82-136.

[3] Gevrey, M., *Démonstration du théorème de Picard-Bernstein par la méthode des contours successifs; prolongement analytique*, Bulletin des Sciences Mathématiques, Volume 50, 1926, pp. 113-128.

[4] Giraud, G., *Équations de type elliptique*, Annales Scientifiques de l'École Normale Supérieure, Paris, Volume 47, 1930, pp. 197-266. References to previous work by the author may be found here.

[5] Hopf, E., *Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung*, Mathematische Zeitschrift, Volume 34, No. 2, 1931, pp. 191-233.

[6] Hopf, E., *Elementare Bermerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte der Preussischen Akademie der Wissenschafter, Volume 19, 1927, pp. 147-152.

[7] Leray, J., *Majoration des dérivées secondes des solutions d'un problème de Dirichlet*, Journal de Mathématiques Pures et Appliquées, Volume 17, No. 9, 1938, pp. 89-104.

[8] Leray, J., *Discussion d'un problème de Dirichlet*. Journal de Mathématiques Pures et Appliquées, Volume 18, No. 9, 1939, pp. 249-284.

[9] Leray, J. and Schauder, J., *Topologie et équations fonctionelles*, Annales Scientifiques de l'École Normale Supérieure, Paris, Volume 51, No. 3, 1934, pp. 45-78. (See sections 19-22).

[10] Lewy, H., *Neuer Beweis des analytischen Charakters der Lösungen elliptischer Differentialgleichungen*, Mathematische Annalen, Volume 101, 1929, pp. 609–619.

[11] Lewy, H., *A priori limitations for solutions of Monge-Ampère equations I*, Transactions of the American Mathematical Society, Volume 37, 1935; pp. 417–434; II, volume 41, 1937, pp. 365–374.

[12] Lichtenstein, L., *Neure Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, article in Encyklöpadie der Wissenschaften, Volume II, Part 3, Second Half, pp. 1277–1334.

[13] Morrey, C. B., *On the solutions of quasilinear elliptic partial differential equations*, Transactions of the American Mathematical Society, Volume 43, 1938, pp. 126–166.

[14] Morrey, C. B., *Multiple Integral Problems in the Calculus of Variations and Related Topics*, University of California Publications in Mathematics, new series, Volume I, 1943, Chapters 2 and 3.

[15] Petrovsky, I. G., *Sur l'analyticité des solutions des systèmes d'équations différentielles*, Recueil Math. (Matematiceskii Sbornik), New Series, Volume 5, No. 47, 1939, pp. 3–70.

[16] Rado, T., *Geometrische Betrachtungen über zweidimensionale reguläre Variationsprobleme*, Acta Literarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephine, Sectio Scientiarum Mathematicarum, Szeged, 1924–1926, pp. 228–253.

[17] Schauder, J., *Der Fixpunktsatz in Funktionalräumen*, Studia Mathematica, Volume II, Fascicle IV, 1930, pp. 171–180.

[18] Schauder, J., *Über das Dirichletsche Problem im Grossen für nichtlineare elliptische Differentialgleichungen*, Mathematische Zeitschrift, Volume 34, No. 4, 1933, pp. 623–634.

[19] Schauder, J., *Über lineare elliptische Differentialgleichungen zweiter Ordnung*, Mathematische Zeitschrift, Volume 38, No. 2, 1934, pp. 257–282.

[20] Shiffman, M., *Differentiability and analyticity of solutions of double integral variational problems*, Annals of Mathematics, Volume 48, No. 2, 1947, pp. 274–284.

[21] von Neumann, J., *Über einen Hilfssatz der Variationsrechnung*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, Volume 8, 1931, pp. 28–31.

Communications on
PURE AND APPLIED MATHEMATICS

This journal publishes papers originating from and solicited by the Institute of Mathematical Science of New York University. It is devoted mainly to contributions in the fields of applied mathematics and mathematical physics, and mathematical analysis.

To Appear in the Next Issue

Uniqueness in Cauchy Problems for Elliptic Systems of Equations

by AVRON DOUGLIS

Oblique Water Entry of a Wedge

by P. R. GARABEDIAN

A Function-Theoretic Approach to Elliptic Systems of Equations in Two Variables

by AVRON DOUGLIS

Nonlinear Hyperbolic Equations

by PETER D. LAX

Flows through Nozzles and Related Problems of Cylindrical and Spherical Waves

by YU WHY CHEN

INTERSCIENCE PUBLISHERS, INC.

JUN 11 1953

DIFFERENTIAL AND INTEGRAL CALCULUS

By R. COURANT, *Institute for Mathematics and Mechanics, New York University.* Translated by J. E. McSHANE, Professor of Mathematics, University of Virginia.
IN TWO VOLUMES.
Volume I: 1937. Second edition, revised, 630 pages, 136 illus. $6.00
Volume II: 1936. 692 pages, 112 illus. $7.50

"The two volumes of Courant's 'Differential and Integral Calculus' form a distinctive treatise of high quality . . . the style is lively . . . It is certain that all American mathematicians will feel grateful to the author, Prof. Courant, and also to the translator, Prof. McShane, for their cooperation in making this excellent textbook available to our mathematical public." GEORGE D. BIRKHOFF in *Science*

PURE AND APPLIED MATHEMATICS

A Series of Texts and Monographs

Edited by H. BOHR, R. COURANT and J. J. STOKER

Volume I:

SUPERSONIC FLOW AND SHOCK WAVES

By R. COURANT and K. O. FRIEDRICHS, *Institute for Mathematics and Mechanics, New York University.* 1948. 480 pages, 216 illus. 6 x 9. $7.50

"This excellent work, which definitely surpasses previous textbooks on the subject in comprehensiveness and thoroughness of treatment of the fundamental concepts will doubtless be . . . most helpful to engineers, physicists, and mathematicians alike, and it is safe to say that no one who has interests in the field of gas dynamics, or indeed in any other branch of non-linear wave propagation, can afford to be without access to it." *Philosophical Magazine.*

Volume II:

NONLINEAR VIBRATIONS IN MECHANICAL AND ELECTRICAL SYSTEMS

By J. J. STOKER, *Institute for Mathematics and Mechanics, New York University.* 1950. 6 x 9. 294 pages, 91 illus. $6.00

"*Nonlinear Vibrations* is exactly the type of text which will appeal to the engineer who, today, must struggle with so many nonlinear phenomena produced by non-cooperative nature." Stephen J. Zand in *Aeronautical Engineering Review*

Volume III:

DIRICHLET'S PRINCIPLE, CONFORMAL MAPPING, AND MINIMAL SURFACES

By R. COURANT, *Institute for Mathematics and Mechanics, New York University.* With an Appendix by M. Schiffer, Princeton University and University of Jerusalem. 1950. 6 x 9. 344 pages, 68 illus. $6.50

"The author presents the modern development of the Dirichlet principle in a manner which is simple, yet independent of the advanced notions of Hilbert space. Physical interpretations and motivations are stressed throughout, and proofs are given with clarity and detail, yet with sufficient assumption of an elementary knowledge to attract the reader who is chiefly interested in research in this field."
P. R. Garabedian in *Mathematical Review*

INTERSCIENCE PUBLISHERS, INC.
250 Fifth Ave., New York 1, N. Y.