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ON THE NON-STEADY MOTIONS OF A RIGID BODY IN AN IDEAL FLUID

by

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On the Non-Steady Motions of a Rigid Body in an Ideal Fluid[1]

By G. W. Morgan

Introduction and Summary. If a rigid body moves in vacuum under the action of a single external force which has a constant direction but may vary in magnitude as well as change sign and whose line of action passes through the center of mass of the body, then the body will maintain a pure translatory motion parallel to the direction of the force. If the body is surrounded by a fluid which is assumed to be non-viscous and incompressible, then such a motion will, in general, no longer be possible. The following question arises: Is there a point such that, if the line of action of a force having constant direction passes through this point, the body will move in a straight line parallel to the force without rotation? The investigation presented below gives the following answers to this question.

a) There is no such point in the most general case.

b) If the shape of the body satisfies certain conditions, then there exist three axes, each associated with the body and with one of three directions relative to the body, (the so-called directions of permanent translation)*, as well as with the density of the fluid, such

1. The results in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.

* Lamb. Hydrodynamics, Ch. VI.

that, if a force of constant or varying magnitude acts on the body with its line of action along one of the axes and if there are no other external forces, the body will move with pure translation parallel to the force and the axis. The conditions which the body shape must satisfy are tantamount to certain symmetry requirements.

c) In the case of plane motion (without circulation), i.e., if the body is an infinite cylinder with its generators perpendicular to its direction of motion, there exists a unique point (for a fixed fluid density ρ) such that a body, whose motion is due entirely to an external force which acts in the direction of one of the two axes of permanent translation with its line of action passing through the point, will move without rotation parallel to the force. This point might be referred to as the "apparent center of mass" of the body corresponding to the given fluid.

d) For a body of revolution there also exists an apparent center of mass.

Inasmuch as these considerations concern a problem in classical hydrodynamics, one might expect to find them in the standard literature on the subject, but, to his surprise, the author was unable to do so.

Our plan of procedure is to set up the expressions for the kinetic energy of the body and the fluid and then to investigate the equations of motion of the body in terms of the kinetic energies.

<u>Kinetic Energy of the Body</u>. Using vector notation, the motion of a body can be described in terms of the translational velocity $\underline{U}$ of a fixed point O of the body and rotation with angular velocity $\underline{\omega}$ about that point. Then the kinetic energy of the body, T_B, is given by:

$$T_B = \int_B \sigma[\underline{U} + \underline{\omega} \times \underline{r}] \cdot [\underline{U} + \underline{\omega} \times \underline{r}]\, d\tau \tag{1}$$

or

$$T_B = \underline{U} \cdot \underline{U} \int_B \sigma d\tau + 2\underline{U} \cdot \int_B \sigma\, \underline{\omega} \times \underline{r}\, d\tau + \int_B (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r}) d\tau \tag{2}$$

where σ = density of the body,

$\underline{r}$ = displacement vector of a point with respect to O,

and the volume integration extends over the entire body B.

Consider a system of rectangular Cartesian axes fixed to the body with origin at O. Let the components of translational and angular velocity resolved along these axes be U_1, U_2, U_3, and ω_1, ω_2, ω_3, respectively. It is convenient to introduce tensor notation and to write U_i, ω_i, x_i for $\underline{U}$, $\underline{\omega}$, $\underline{r}$ respectively.

Then, writing (2) in terms of tensors, we have

$$T_B = U_i U_i \int_B \sigma d\tau + 2U_i \varepsilon_{ijk} \omega_j \int_B \sigma x_k d\tau + \int_B \sigma(\varepsilon_{ijk}\omega_j x_k \varepsilon_{i\ell n}\omega_\ell x_n) d\tau \tag{3}$$

where ε_{ijk} is the alternating tensor. We transform the integrand of the last integral as follows:

$$\varepsilon_{ijk}\varepsilon_{i\ell n}x_k x_n \omega_j \omega_\ell = (\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell})x_k x_n \omega_j \omega_\ell =$$

$$= (\delta_{j\ell}x_k x_k - x_j x_\ell)\omega_j \omega_\ell .$$

Let

$$\int_B \sigma(\delta_{j\ell}x_k x_k - x_j x_\ell)d\tau = I_{j\ell}$$

where $I_{j\ell}$ is the inertia tensor of the body. Then (3) can be written

$$T_B = U_i U_i \left(\int_B \sigma d\tau\right) + U_i \omega_j \left(\varepsilon_{ijk}\int_B \sigma x_k d\tau\right) + \\ + \omega_j U_i \left(\varepsilon_{ijk}\int_B \sigma x_k d\tau\right) + \omega_i \omega_j I_{ij}. \tag{4}$$

We note that this expression involves ten independent constants:

$$\int_B \sigma d\tau = M, \quad \text{the mass of the body},$$

$$\int_B \sigma x_k d\tau = M\bar{x}_k, \qquad k = 1, 2, 3$$

where $\bar{x}_k$ are the coordinates of the center of mass, and finally the six independent components of the symmetric inertia tensor I_{ij}.

At this point it is convenient to introduce a new notation.

Let

$$V_1 = U_1, \quad V_2 = U_2, \quad V_3 = U_3$$
$$V_4 = \omega_1, \quad V_5 = \omega_2, \quad V_6 = \omega_3. \tag{5}$$

By V_α we shall mean any of the six components V_1 to V_6 with α ranging from 1 to 6. We use Greek subscripts to differentiate the present notation from tensor notation where the subscripts i, j, etc. take on values 1, 2 or 3 only.

Also, supposing the summations implied in expression (4) to be carried out and the U_i and ω_i to be replaced by the appropriate V_α, let $M_{\alpha\beta}$ be the coefficient of the term in $V_\alpha V_\beta$. The 36 quantities $M_{\alpha\beta}$ then have the following values:

$$
\begin{aligned}
&M_{11} = M_{22} = M_{33} = M \\
&M_{12} = M_{21} = M_{13} = M_{31} = M_{23} = M_{32} = 0 \\
&M_{14} = M_{41} = M_{25} = M_{52} = M_{36} = M_{63} = 0 \\
&M_{15} = M_{51} = -M_{24} = -M_{42} = M\bar{x}_3 \\
&M_{16} = M_{61} = -M_{34} = -M_{43} = -M\bar{x}_2 \\
&M_{26} = M_{62} = -M_{35} = -M_{53} = M\bar{x}_1 \\
&M_{44} = I_{11}; \quad M_{55} = I_{22}; \quad M_{66} = I_{33} \\
&M_{45} = I_{12}; \quad M_{54} = I_{21}; \quad \text{hence} \quad M_{45} = M_{54} \\
&M_{46} = I_{13}; \quad M_{64} = I_{31}; \quad \text{hence} \quad M_{46} = M_{64} \\
&M_{56} = I_{23}; \quad M_{65} = I_{32}; \quad \text{hence} \quad M_{56} = M_{65}.
\end{aligned}
\tag{6}
$$

We can now write expression (4) for the kinetic energy in the compact form

$$T_B = \sum_{\alpha,\beta=1}^{6} V_\alpha V_\beta M_{\alpha\beta} \tag{7}$$

Kinetic Energy of the Fluid.

Since the motion of the fluid is assumed to be entirely due to that of the solid, it is irrotational. Its velocity potential Φ is of the form

$$\Phi = \sum_{\alpha=1}^{6} V_\alpha \varphi_\alpha \tag{8}$$

where the φ_α are functions of the coordinates x_1, x_2, x_3 depending solely on the shape and position of the solid surface relative to the coordinate axes fixed in the body. They represent the velocity potentials due to the motion of the body with unit translational or rotational velocity with respect to the appropriate axes.* The kinetic energy of the fluid, T_F, is given by

$$T_F = \rho \int_S \Phi \frac{\partial \Phi}{\partial n} \, dS \tag{9}$$

where ρ is the fluid density, $\frac{\partial}{\partial n}$ denotes differentiation with respect to the outwardly directed normal to the surface S, and the integration extends over the surface S of the body.

Substitution of (3) into (9) gives

$$T_F = \rho \int_S \left(\sum_\alpha V_\alpha \varphi_\alpha \sum_\beta V_\beta \frac{\partial \varphi_\beta}{\partial n} \right) dS$$

$$= \sum_{\alpha,\beta} \left(V_\alpha V_\beta \rho \int_S \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n} \, dS \right). \tag{10}$$

Each integral $\displaystyle\int_S \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n} \, dS$

---

* Lamb, loc. cit.

is a constant determined exclusively by the shape and position of the body surface S relative to the coordinate axes. Hence setting

$$\rho \int_S \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n} dS = F_{\alpha\beta}, \tag{11}$$

we can write (10) in the form

$$T_F = \sum_{\alpha,\beta=1}^{6} V_\alpha V_\beta F_{\alpha\beta}. \tag{12}$$

Since $\nabla^2 \varphi_\alpha = 0$ for all α,

$$\int_S \varphi_\alpha \frac{\partial \varphi_\beta}{\partial n} dS = \int_S \varphi_\beta \frac{\partial \varphi_\alpha}{\partial n} dS$$

so that $F_{\alpha\beta} = F_{\beta\alpha}$ and hence equation (12) involves 21 independent constants.

<u>Equations of Motion of the Body.</u>

The total kinetic energy T of fluid and body combined is

$$T = T_B + T_F = \sum_{\alpha,\beta} V_\alpha V_\beta (M_{\alpha\beta} + F_{\alpha\beta}). \tag{13}$$

The equations of motion of the body in terms of T are:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial T}{\partial V_1} - V_6 \frac{\partial T}{\partial V_2} + V_5 \frac{\partial T}{\partial V_3} = X_1 \\
&\frac{d}{dt}\frac{\partial T}{\partial V_2} - V_4 \frac{\partial T}{\partial V_3} + V_6 \frac{\partial T}{\partial V_1} = X_2 \\
&\frac{d}{dt}\frac{\partial T}{\partial V_3} - V_5 \frac{\partial T}{\partial V_4} + V_4 \frac{\partial T}{\partial V_2} = X_3 \\
&\frac{d}{dt}\frac{\partial T}{\partial V_4} - V_3 \frac{\partial T}{\partial V_2} + V_2 \frac{\partial T}{\partial V_3} - V_6 \frac{\partial T}{\partial V_5} + V_5 \frac{\partial T}{\partial V_6} = X_4 \\
&\frac{d}{dt}\frac{\partial T}{\partial V_5} - V_1 \frac{\partial T}{\partial V_3} + V_3 \frac{\partial T}{\partial V_1} - V_4 \frac{\partial T}{\partial V_6} + V_6 \frac{\partial T}{\partial V_4} = X_5 \\
&\frac{d}{dt}\frac{\partial T}{\partial V_6} - V_2 \frac{\partial T}{\partial V_1} + V_1 \frac{\partial T}{\partial V_2} - V_5 \frac{\partial T}{\partial V_4} + V_4 \frac{\partial T}{\partial V_5} = X_6
\end{aligned} \tag{14}$$

where X_1, X_2, X_3 are the components of the external forces and X_4, X_5, X_6 are the components of the external moments with respect to O that are applied to the body. (By external forces and moments we mean forces and moments other than those due to the fluid pressures.)

Rectilinear Motion.

We now choose a special orientation for our coordinate axes, namely that of the three mutually perpendicular axes of "permanent translation" of the body. These have the property that if the body is set in motion parallel to one of them, without rotation and in the absence of external forces, it will continue to move in this manner; i.e., for motion with constant velocity in one of these directions the fluid exerts no force or moment on the body. We shall now proceed to examine the external forces and moments necessary for accelerated motion of the body in one of these directions.

It can be shown that if the directions of permanent translation are chosen as coordinate directions, the constants

$$F_{12} = F_{13} = F_{23} = F_{21} = F_{31} = F_{32} = 0. \qquad (15)$$

In the following we shall always consider this choice to have been made.

Motion with Velocity $V_1(t)$.

Let us consider first the case when $V_1 = V_1(t)$ and all other $V_\alpha = 0$ for all time. Then the equations of motion (14) together with the expression for the kinetic energy (13) give:

$$
\begin{aligned}
2\dot{V}_1 A_{11} &= X_1 \\
2\dot{V}_1 A_{12} &= X_2 \\
2\dot{V}_1 A_{13} &= X_3 \\
2\dot{V}_1 A_{14} &= X_4 \\
2\dot{V}_1 A_{15} - 2V_1^2 A_{13} &= X_5 \\
2\dot{V}_1 A_{16} + 2V_1^2 A_{12} &= X_6
\end{aligned}
\tag{16}
$$

Hence from (6) and (15)

$$
\begin{aligned}
&X_1 = 2\dot{V}_1 A_{11}; \quad X_2 = X_3 = 0 \\
&X_4 = 2\dot{V}_1 A_{14}; \quad X_5 = 2\dot{V}_1 A_{15}; \quad X_6 = 2\dot{V}_1 A_{16}.
\end{aligned}
\tag{17}
$$

Expressions (17) represent the system of forces X_1, X_2, X_3 whose line of action passes through O, and the system of moments X_4, X_5, X_6 about O, which must be applied to the body so that it may move in the prescribed manner. Our object is to ascertain if there exists a point $O^{(1)}$ such that, if the line of action of the forces X_1, X_2, X_3 passes through it instead of through O, the external moments $X_4^{(1)}$, $X_5^{(1)}$, $X_6^{(1)}$ with respect to $O^{(1)}$ will vanish. If such a point exists, then, under the sole action of the external force X_1 applied through $O^{(1)}$, (we may now conveniently call this force $X_1^{(1)}$), the body will always move without rotation and in the direction of the x_1 axis.

To investigate this question we refer all quantities to a new coordinate system whose origin is at a point $O^{(1)}$ and whose axes are parallel to those of the original system.

Let the velocities of the body referred to this new point be $V_\alpha^{(1)}$; and let $d_1^{(1)}$, $d_2^{(1)}$, $d_3^{(1)}$ be the components of the displacement of $O^{(1)}$ from O.

Then

$$
\begin{aligned}
&V_4 = V_4^{(1)}; \quad V_5 = V_5^{(1)}; \quad V_6 = V_6^{(1)} \\
&V_1 = V_1^{(1)} + V_6^{(1)}\, d_2^{(1)} - V_5^{(1)} \\
&V_2 = V_2^{(1)} + V_4^{(1)}\, d_3^{(1)} - V_6^{(1)}\, d_1^{(1)} \\
&V_3 = V_3^{(1)} + V_5^{(1)}\, d_1^{(1)} - V_4^{(1)}\, d_2^{(1)}
\end{aligned}
\tag{18}
$$

If we substitute expressions (18) into the formula (13) for the kinetic energy and collect coefficients of $V_\alpha^{(1)}\, V_\beta^{(1)}$, which we call $A_{\alpha\beta}^{(1)}$, we obtain

$$
\begin{aligned}
A_{14}^{(1)} &= A_{14} + A_{12}\, d_3^{(1)} - A_{13}\, d_2^{(1)} \\
A_{15}^{(1)} &= A_{15} + A_{13}\, d_1^{(1)} - A_{11}\, d_3^{(1)} \\
A_{16}^{(1)} &= A_{16} + A_{11}\, d_2^{(1)} - A_{12}\, d_1^{(1)}.
\end{aligned}
\tag{19}
$$

The remaining coefficients are not of interest at the moment.

If we now rewrite the equations of motion with respect to the $V_\alpha^{(1)}$ and $O^{(1)}$ we obtain equations identical with (17) except that all quantities will now have superscripts (1). Hence the external moments about $O^{(1)}$ can vanish only if

$$A_{14}^{(1)} = A_{15}^{(1)} = A_{16}^{(1)} = 0.$$

Using (19), (15) and (6) this means we must have:

$$
\begin{aligned}
&F_{14} = F_{14}^{(1)} = 0 \\
&F_{15} + M_{15} - (F_{11} + M_{11})\, d_3^{(1)} = 0 \\
&F_{16} + M_{16} + (F_{11} + M_{11})\, d_2^{(1)} = 0.
\end{aligned}
\tag{20}
$$

Since F_{14} does not vanish in general, equations (20) cannot generally be satisfied and hence there exists no point with respect to which the moment of fluid pressures vanishes for accelerated motion in the direction of the x_1 axis. It is clear that analogous conclusions will be reached for motion along the two other axes.

We may note that this result could have been deduced directly from the equations (17) referring to the original system. The components of the external force which can contribute to the moment $X_4^{(1)}$ are X_1 and X_2; but these vanish. Hence we must have $X_4 = X_4^{(1)}$, i.e., the moment is the same for any reference point and is therefore due to a pure couple. Thus a change in the point of application of the external force can have no effect on X_4.

Returning to equations (20) we see that if F_{14} happens to vanish for a particular body, then all the equations (20) can be satisfied by choosing $O^{(1)}$ appropriately. Using (6) we then have:

$$d_3^{(1)} = \frac{F_{15} + M\bar{x}_3}{F_{11} + M}$$
$$d_2^{(1)} = -\frac{F_{16} - M\bar{x}_2}{F_{11} + M}. \qquad (21)$$

Hence in this case we can find a point $O^{(1)}$, or rather an axis $x_1^{(1)}$, (since $d_1^{(1)}$ is arbitrary), such that under the sole action of a force $X_1^{(1)}$ along this axis, the body will always move parallel to this axis and without rotation. We might call such an axis, an "axis of rectilinear translation".

The condition that $F_{14} = 0$ imposes a restriction on the body shape and so might be regarded as a symmetry requirement.

If the original origin O is taken at the center of mass of the body then the terms $M\bar{x}_3$ and $M\bar{x}_2$ vanish in (21).

Motion with Velocity $V_2(t)$.

If we return to our original system of axes with origin at O and consider the case when all V_α except V_2 vanish for all time, we obtain the following equations of motion analogous to (16).

$$\begin{aligned} 2\dot{V}_2 A_{12} &= X_1 \\ 2\dot{V}_2 A_{22} &= X_2 \\ 2\dot{V}_2 A_{23} &= X_3 \\ 2\dot{V}_2 A_{24} + 2V_2^2 A_{23} &= X_4 \\ 2\dot{V}_2 A_{25} &= X_5 \\ 2\dot{V}_2 A_{26} - 2V_2^2 A_{12} &= X_6, \end{aligned} \tag{22}$$

and hence from (6) and (15)

$$\begin{aligned} &X_1 = X_3 = C; \quad X_2 = 2\dot{V}_2 A_{22}; \\ &X_4 = 2\dot{V}_2 A_{24}; \quad X_5 = 2\dot{V}_2 A_{25}; \quad X_6 = 2\dot{V}_2 A_{26}. \end{aligned} \tag{23}$$

As before, we transform to a new system with origin at $O^{(2)}$. The relations between the velocities are identical with (18) except that we now have superscripts (2). The pertinent quantities $A_{\alpha\beta}^{(2)}$ become:

$$A_{24}^{(2)} = A_{24} + A_{22}\, d_3^{(2)} - A_{23}\, d_2^{(2)}$$

$$A_{25}^{(2)} = A_{25} + A_{23}\, d_1^{(2)} - A_{21}\, d_3^{(2)} \qquad (24)$$

$$A_{26}^{(2)} = A_{26} + A_{21}\, d_2^{(2)} - A_{22}\, d_1^{(2)} .$$

Again, the equations of motion with respect to the $V_\alpha^{(2)}$ and $O^{(2)}$ will be the same as those with respect to the V_α and O (equations (23)), except that all quantities will have superscripts (2).

The external moments about $O^{(2)}$ will vanish if

$$A_{24}^{(2)} = A_{25}^{(2)} = A_{26}^{(2)} = 0,$$

which, by (24), (15) and (6) becomes:

$$F_{24} = M_{24} + (F_{22} + M_{11})\, d_3^{(2)} = 0$$

$$F_{25} = 0 \qquad (25)$$

$$F_{26} + M_{26} - (F_{22} + M_{11})\, d_1^{(2)} = 0.$$

As expected, these equations have no solution in the general case since F_{25} is not generally zero.

If F_{25} happens to be zero, the equations (25) are satisfied if:

$$d_3^{(2)} = - \frac{F_{24} - M\bar{x}_3}{F_{22} + M}$$

and (26)

$$d_1^{(2)} = \frac{F_{26} + M\bar{x}_1}{F_{22} + M} .$$

Hence there exists an axis $x_2^{(2)}$ analogous to the axis $x_1^{(1)}$ found previously. We note, however, that the two axes will

not, in general, intersect since $d_3^{(1)} \neq d_3^{(2)}$ except under special circumstances.

Since an analogous result will be obtained for motion with velocity V_3 we can sum up our results as follows:

(i) In general, rectilinear translational motion under the sole action of one external force is not possible.

(ii) If the body shape satisfies a certain symmetry condition (e.g., $F_{14} = 0$ when referred to coordinate axes parallel to the axes of permanent translation), then there exists an axis of rectilinear translation for motion with velocity V_1. Similarly, if $F_{25} = 0$ or $F_{36} = 0$, there exist axes of rectilinear translation for motion with velocity V_2 or V_3, respectively.

(iii) In general no two of these axes, if more than one exists, will intersect.

Plane Motion Without Circulation.

If the body is an infinite cylinder with its generators parallel to the x_3 axis which moves with velocity components V_1, V_2 and V_6 only, then the fluid motion will take place in planes parallel to the x_1, x_2 plane and the two axes of permanent translation will lie in this plane. As before we choose our coordinate system in the direction of these axes. For such plane motion all coefficients $A_{\alpha\beta}$ in the expression (13) for the kinetic energy having one or both subscripts equal to 3, 4 or 5, vanish.

Considering motion with velocity V_1 only, we obtain equations (17) with $A_{14} = A_{15} = 0$ and hence $X_4 = X_5 = 0$. The transformation to a new coordinate system with origin $O^{(1)}$ yields one equation for $d_2^{(1)}$ which will make $A_{16}^{(1)} = 0$. It is the same

as the second of equations (21). This determines one axis of rectilinear translation.

Similar arguments hold for motion with velocity $V_2(t)$ and they lead to the second of equations (26) for $d_1^{(2)}$ which will make $A_{26}^{(2)} = 0$. This determines the second axis of rectilinear motion.

Since both axes lie in the plane of motion, they intersect at a point P, say, with coordinates $\bar{x}_1^{(a)}$ and $\bar{x}_2^{(a)}$. If the origin of our first coordinate system is taken at the center of mass of the body, then $\bar{x}_1^{(a)}$ and $\bar{x}_2^{(a)}$ are given by the equations for $d_2^{(1)}$ and $d_1^{(2)}$, respectively, with $\bar{x}_1 = \bar{x}_2 = 0$:

$$\begin{aligned} x_1^{(a)} &= \frac{F_{26}}{F_{22} + M} \\ \bar{x}_2^{(a)} &= - \frac{F_{16}}{F_{11} + M} . \end{aligned} \tag{27}$$

We might call this point the "apparent center of mass of the body", keeping in mind, however, that its position also depends on the density of the fluid which appears in the constants $F_{\alpha\beta}$.

It is interesting to consider pure rotation about this point with velocity $V_6(t)$. The equations of motion are found to be:

$$\begin{aligned} 2A_{16}^{(a)} \dot{V}_6 - 2V_6 A_{26}^{(a)} &= X_1^{(a)} \\ 2A_{26}^{(a)} \dot{V}_6 + 2V_6^2 A_{16}^{(a)} &= X_2^{(a)} \\ X_3 = X_4 = X_5 &= 0 \\ 2A_{66}^{(a)} \dot{V}_6 &= X_6^{(a)} , \end{aligned} \tag{28}$$

where the superscript (a) means that we are referring all quantities to a coordinate system whose axes are the axes of rectilinear motion and whose origin is therefore at $\bar{x}_1^{(a)}$, $\bar{x}_2^{(a)}$. Since $A_{16}^{(a)} = A_{26}^{(a)} = 0$ the only external force or moment on the body is a moment about the origin. Thus the point $\bar{x}_1^{(a)}$, $\bar{x}_2^{(a)}$ also acts as an apparent center of mass in the case of pure rotation about this point.

It is important to note that the motions and corresponding forces and moments considered above cannot be simply superimposed. A translation of the apparent center of mass, for example, with velocity components V_1 and V_2 but without rotation, cannot occur without the action of an external moment about the apparent center of mass. This is due to the non-linearity of the problem. The analogy between the apparent center of mass for motion of a body in a fluid and the center of mass is restricted to three particular motions, the two translations of the body parallel to the axes of permanent (or rectilinear) translation, and rotation about the apparent center of mass.

Body of Revolution.

The axis of revolution is a direction of permanent translation. Let it be the x_1 axis. For motion in the x_1 direction we consider the equations (20). For a body of revolution F_{14} must always be zero because changing the sign of V_4 in the expression for the kinetic energy must leave the latter unchanged. Similarly $F_{15} = F_{16} = 0$; also $M_{15} = M\bar{x}_3 = 0$ and $M_{16} = - M\bar{x}_2 = 0$ since the center of mass lies on the axis of revolution; hence

$d_3^{(1)} = d_2^{(1)} = 0$. This gives the obvious result that the axis of rotation is an axis of rectilinear translation.

From symmetry considerations it is clear that the choice of directions for the axes x_2 and x_3 is immaterial. For motion with velocity V_2, say, we must consider equations (25). Applying symmetry considerations to the expression for T, as was done above, we find that $F_{24} = F_{25} = 0$. Hence

$$A_{24}^{(2)} = M_{24} + (F_{22} + M_{11})\, d_3^{(2)} = 0$$
$$A_{26}^{(2)} = F_{26} + M_{26} - (F_{22} + M_{11})\, d_1^{(2)} = 0. \tag{29}$$

But

$$M_{24} = - M\bar{x}_3 = 0.$$

Thus $d_3^{(2)} = 0$, and there remains

$$d_1^{(2)} = \frac{F_{26} + M\bar{x}_1}{F_{22} + M} \tag{30}$$

which defines an axis of rectilinear translation.

For motion with velocity V_3 we obtain in the same manner $d_2^{(3)} = 0$ and

$$d_1^{(3)} = - \frac{F_{35} - M\bar{x}_1}{F_{33} + M}\,. \tag{31}$$

From the symmetry of the body it is seen that $F_{22} = F_{33}$ and $F_{26} = - F_{35}$. Hence $d_1^{(2)} = d_1^{(3)}$ as was to be expected from our assertion that the choice of the directions x_2 and x_3 was immaterial.

Thus, as in the case of plane motion, there exists a center of apparent mass, but here there are infinitely many axes

of rectilinear motion: the axis of revolution and any axis perpendicular to the latter and passing through the apparent center of mass.

Let us consider a motion of pure rotation about an axis of rectilinear motion. Clearly, for motion with $V_4(t)$, i.e., rotation about the axis of revolution, the only force or moment is

$$X_6^{(a)} = M_{44}^{(a)} \dot{V}_4.$$

For motion with velocity V_6, say, about the appropriate axis of rectilinear motion we have:

$$\begin{aligned}
2A_{16}^{(a)} \dot{V}_6 - 2V_6^2 A_{26}^{(a)} &= X_1^{(a)} \\
2A_{26}^{(a)} \dot{V}_6 + 2V_6^2 A_{16}^{(a)} &= X_2^{(a)} \\
2A_{36}^{(a)} \dot{V}_6 &= X_3^{(a)} \\
2A_{46}^{(a)} \dot{V}_6 - 2V_6^2 A_{56}^{(a)} &= X_4^{(a)} \\
2A_{56}^{(a)} \dot{V}_6 + 2V_6^2 A_{46}^{(a)} &= X_5^{(a)} \\
2A_{66}^{(a)} \dot{V}_6 &= X_6^{(a)}.
\end{aligned} \tag{32}$$

But $A_{16}^{(a)} = A_{26}^{(a)} = 0$ and $A_{36}^{(a)}$, $A_{46}^{(a)}$, $A_{56}^{(a)}$ vanish from symmetry considerations. Hence all forces and moments with the exception of $X_6^{(a)}$ vanish. Thus the concept of the apparent center of mass for a body of revolution is also valid for pure rotation about an axis of rectilinear motion.

As in the plane case, however, simple superposition of the special motions and forces discussed is not permissible.

It should be noted that, even if all assumptions made in this analysis are sufficiently well justified, some of the special motions discussed cannot be expected to be observed in nature because they are unstable when subjected to small disturbances.

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