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NAVORD REPORT 2725

A METHOD FOR THE NUMERICAL SOLUTION OF A HEAT CONDUCTION PROBLEM

31 December 1952

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NAVORD Report 2725
Aeroballistic Research Report No. 86

A METHOD FOR THE NUMERICAL SOLUTION OF A HEAT CONDUCTION PROBLEM

Prepared by

R. P. Eddy

ABSTRACT: A scheme, due to J. von Neumann, for the numerical integration of parabolic or hyperbolic partial differential equations is here set up for use in solving a particular problem in heat conduction. A crude stability analysis is given, showing that the effects of round-off errors tend to be damped out regardless of the size of the time step — a situation which does not hold true for some better known integration schemes.

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NAVORD Report 2725 31 December 1952

This report contains a description of a scheme for numerical integration of partial differential equations of parabolic or of hyperbolic type, due to J. von Neumann. This report has been organized under Project NR-044-003, work in Numerical Analysis, sponsored by the Office of Naval Research.

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A METHOD FOR THE NUMERICAL SOLUTION OF A HEAT CONDUCTION PROBLEM

1. Introduction

In the course of development of a certain high velocity projectile it became desirable to estimate the temperature of the projectile's outer shell or "skin," during the course of flight. One way of accomplishing this was to obtain a numerical solution of the partial differential equation which governs heat flow in an infinite plate, using appropriate boundary conditions. This equation was non linear due to the fact that the thermal properties of steel vary appreciably over the temperature range encountered. At the inner surface the boundary condition was taken to be no transfer to heat across the surface, while at the outer surface the boundary condition took into account heat transfer both by conduction to or from the boundary layer and by radiation into space. (The latter effect turned out to be almost negligible.)

A scheme for numerical integration suggested by Dr. John von Neumann of the Institute for Advanced Study was used for part of the calculations and will be described here. It is an "implicit method" where in the values of the dependent variable on the new time step are expressed in terms of each other and must be obtained by solving a simple system of linear algebraic equations; its chief interest lies in its stability: round-off errors are damped out regardless of the size of the time step.

Numerical results which have been obtained in connection with this study can be provided upon request.

2. Equations

The governing partial differential equation was taken to be

$$\rho C(T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[K(T) \frac{\partial T}{\partial x} \right] \tag{1}$$

with initial condition

$$T(x, 0) = 70^\circ F = 530^\circ R \tag{2}$$

and boundary conditions

$$\left[K(T) \frac{\partial T}{\partial x} \right]_{x=\ell} = 0 \tag{3}$$

$$\left[K(T) \frac{\partial T}{\partial x} \right]_{x=0} = h(t)\left[T(0,t) - G(t) \right] + b\left[(T(0,t))^4 - T_a^4 \right] \tag{4}$$

where the meaning of the symbols is as follows:

- T temperature in degrees Rankine (absolute Fahrenheit)
- t time in hours
- x distance in feet
- K(T) thermal conductivity in BTU/hr. ft. °F
- C(T) heat capacity in BTU/lb. °F
- ρ density in lb/ft^3. (assumed constant)
- h(t) coefficient of heat transfer between gas boundary layer and steel skin
- G(t) temperature of gas boundary layer
- Ta effective black body temperature of space, assumed for simplicity to be constant at 0°F = 460°R.
- b net coefficient of radiation emissivity

The functions h(t) and G(t) had been calculated previously using the best data available for a number of different calculated trajectories.

The functions K(T) and C(T) were approximated by piece-wise linear functions like those in Figure 1,

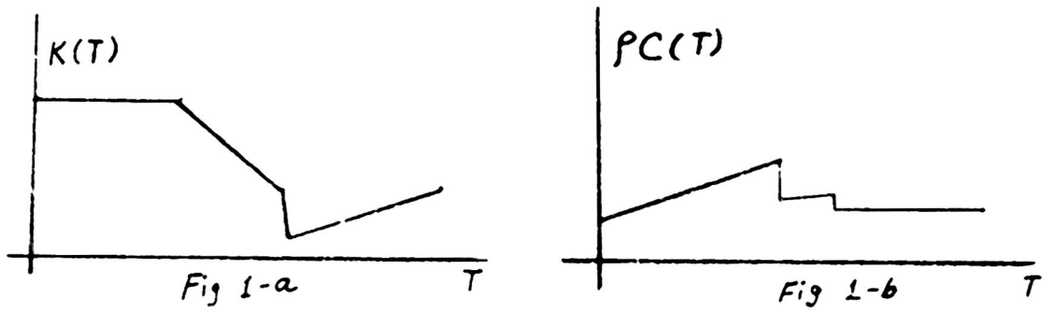

Fig 1-a Fig 1-b

3. Finite difference formulation.

The finite difference mesh used is as shown in Figure 2. For convenience in satisfying boundary conditions, the mesh is laid out so that both boundaries fall

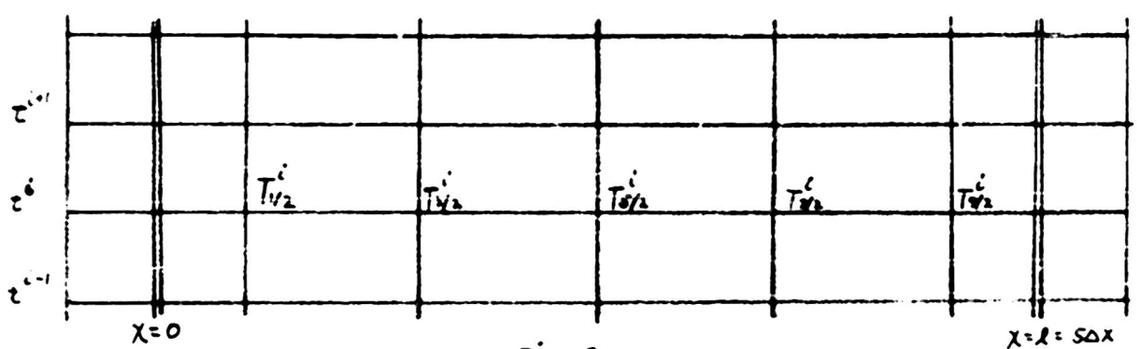

Fig 2

half way between mesh lines. The number of vertical mesh lines (size of Δx) was determined largely by the memory capacity of the computing machine used: the Mark II Relay Calculator at the Naval Proving Ground, Dahlgren, Virginia. A superscript on a symbol refers to the time coordinate, while a subscript refers to the space coordinate. Thus

$$T^{i}_{j-1/2} = T\left((j-1/2)\Delta x,\ i\Delta t\right) \tag{5}$$

The presence of non-integer indices is due partly to the fact that the first mesh line in from the left boundary (Figure 2) lies at $x = 1/2\,\Delta x$ instead of at $x = \Delta x$, and partly to the extensive use of averaging or forming linear combinations of the values of some variable at several mesh points in order to define the value of another variable at an intermediate point. The reason for the latter is, of course, to secure greater accuracy.

The following auxiliary quantities are introduced:

$$\eta^{i+1/2}_{j-1/2} = \tfrac{1}{2}\left(T^{i+1}_{j-1/2} + T^{i}_{j-1/2}\right) \qquad (j=1,2,3,4,5) \tag{6}$$

$$u^{i}_{j} = v^{i}_{j} = K\left(\tfrac{3}{4}\left(T^{i}_{j-1/2} + T^{i}_{j+1/2}\right) - \tfrac{1}{4}\left(T^{i-1}_{j-1/2} + T^{i-1}_{j+1/2}\right)\right) \qquad (j=1,2,3,4) \tag{7}$$

$$u^{i}_{5} = v^{i}_{5} = v^{i}_{0} = 0 \tag{8}$$

$$u^{i}_{0} = (\Delta x)\, h\left(t^{i+1/2}\right) \tag{9}$$

$$w^{i}_{0} = (\Delta x)\left(h\left(t^{i+1/2}\right) G\left(t^{i+1/2}\right) - \left(\eta^{i+1/2}_{1/2}\right)^{4} + \eta_{a}^{4}\right. \tag{10}$$

$$w^{i}_{j} = 0 \qquad (j=1,2,3,4,5) \tag{11}$$

$$\gamma^{i+1/2}_{j-1/2} = \left(2(\Delta x)^{2}/\Delta t\right) \rho C\left(\tfrac{3}{2} T^{i}_{j-1/2} - \tfrac{1}{2} T^{i-1}_{j-1/2}\right) \qquad (j=1,2,3,4,5) \tag{12}$$

With these definitions one has

$$K(T)\frac{\partial T}{\partial x}\Big]^{i+1/2}_{j} = u^{i}_{j}\left(\eta^{i+1/2}_{j+1/2} - \eta^{i+1/2}_{j-1/2}\right)/\Delta x \tag{13}$$

$$\frac{\partial}{\partial x}\left(K(T)\frac{\partial T}{\partial x}\right)\Big]^{i+1/2}_{j-1/2} = \Big\{u^{i}_{j}\left(\eta^{i+1/2}_{j+1/2} - \eta^{i+1/2}_{j-1/2}\right)/\Delta x \tag{14}$$

$$- v^{i}_{j-1}\left(\eta^{i+1/2}_{j-1/2} - \eta^{i+1/2}_{j-3/2}\right)/\Delta x + w^{i}_{j-1}\Big\}/\Delta x$$

$$\rho C(T)\frac{\partial T}{\partial t}\Big]_{j-1/2}^{i+1/2} = \rho C(\eta_{j-1/2}^{i+1/2})\left\{T_{j-1/2}^{i+1} - T_{j-1/2}^{i}\right\}/\Delta t$$

$$= 2\rho C(\eta_{j-1/2}^{i+1/2})\left\{\eta_{j-1/2}^{i+1/2} - T_{j-1/2}^{i}\right\}/\Delta t \tag{15}$$

Substitution of these into (1) leads to the following system of equations

$$\eta_{j-3/2}^{i+1/2} = A_{j-3/2}\,\eta_{j-1/2}^{i+1/2} - B_{j-3/2}\,\eta_{j+1/2}^{i+1/2} - \Gamma_{j-3/2}^{i} \tag{16}$$

where

$$\left.\begin{aligned} A_{j-3/2} &= (\gamma_{j-1/2}^{i+1/2} + u_j^i + v_{j-1}^i)/v_{j-1}^i \\ B_{j-3/2} &= u_j^i / v_{j-1}^i \\ \Gamma_{j-3/2} &= (\gamma_{j-1/2}^{i+1/2}\,T_{j-1/2}^{i} + w_{j-1}^i)/v_{j-1}^i \end{aligned}\right\}\ (j = 2, 3, 4, 5) \tag{17}$$

From (3) and (13) one has $\eta_{11/2} = \eta_{9/2}$ so that (16) allows $\eta_{7/2}$ to be expressed in the form $\eta_{7/2} = \mu_{7/2}\,\eta_{9/2} - \nu_{9/2}$ and, in general

$$\eta_{j-3/2} = \mu_{j-3/2}\,\eta_{9/2} - \nu_{j-3/2} \tag{18}$$

where

$$\left.\begin{aligned} \mu_{j-3/2} &= A_{j-3/2}\,\mu_{j-1/2} - B_{j-3/2}\,\mu_{j+1/2} \\ \nu_{j-3/2} &= A_{j-3/2}\,\nu_{j-1/2} - B_{j-3/2}\,\nu_{j+1/2} + \Gamma_{j-3/2} \end{aligned}\right\}\ (j = 4, 3, 2) \tag{19}$$

when $j = 1$ this recursion scheme breaks down since $v_0 = 0$; however the equation obtained from (16) by multiplying through by v_{j-1}^i, then putting $j = 1$ and $v_0^i = 0$, gives

$$u_0^i\,\eta_{3/2}^{i+1/2} - (\gamma_{1/2}^{i+1/2} + u_1^i)\,\eta_{1/2}^{i+1/2} + \gamma_{1/2}^{i+1/2}\,T_{1/2}^{i} + w_0^i = 0 \tag{20}$$

Elimination of $\eta_{3/2}^{i+1/2}$ and $\eta_{1/2}^{i+1/2}$ by means of (18) with $j = 3$ and $j = 2$, respectively, leaves a single equation in $\eta_{9/2}$ which can be solved immediately. Substitution of this value into the expressions already set up from (18) then gives the values of the remaining $\eta_{j-3/2}^{i+1/2}$ for $j = 5, 4, 3, 2$. From this the desired temperatures are then found with the aid of (6):

$$T_{j-1/2}^{i+1} = 2\,\eta_{j-1/2}^{i+1/2} - T_{j-1/2}^{i}\ . \tag{21}$$

4. Stability Argument

As mentioned in the introduction, one of the interesting features of this integration scheme is that errors due to round-off are damped out no matter how

large Δt is taken — something which is not true of some simpler schemes. The following crude argument, although not a rigorous proof, is of a kind which has proved to be quite useful and fairly accurate in practice.

The first step is to ignore the variability of the coefficients of (1); this leads to a partial difference equation with constant coefficients and homogeneous boundary conditions, which is satisfied by any round-off error function, and which can be solved by separation of variables:

$$\left(T^{i+1}_{j+1/2} - 2\left(1+\frac{1}{2r}\right)T^{i+1}_{j-1/2} + T^{i+1}_{j-3/2}\right) + \left(T^{i}_{j+1/2} - 2\left(1-\frac{1}{2r}\right)T^{i}_{j-1/2} + T^{i}_{j-3/2}\right) = 0 \tag{22}$$

where

$$r = \frac{K\,\Delta t}{2\rho\, C\cdot(\Delta x)^2} \,. \tag{23}$$

Let

$$T^{i}_{j-1/2} = U_j\, V^i \tag{24}$$

where U is a function of x alone and V is a function of t alone. Substituting (24) into (22) and writing $(\lambda-1)/(\lambda+1)$ for the separation constant, one has

$$V^{i+1} = \left(\frac{1-\lambda}{1+\lambda}\right) V^i \tag{25}$$

$$U_{j+1} - 2\left(1-\frac{\lambda}{2r}\right)U_j + U_{j-1} = 0 \,. \tag{26}$$

To solve (26), put

$$1 - \frac{\lambda}{2r} = \cos\varphi \tag{27}$$

$$U_j = s^j \tag{28}$$

which leads to

$$s = e^{\varphi\sqrt{-1}} \tag{29}$$

so that

$$U_j = \alpha\cos j\varphi + \beta\sin j\varphi \,. \tag{30}$$

The boundary conditions then determine a set of values of φ_n such that (30) is an acceptable solution of (26). These, in turn, determine the eigenvalues

$$\lambda_n = 2r(1-\cos\varphi_n) \tag{31}$$

which are never negative, since r is positive. Therefore

$$-1 < \frac{1-\lambda}{1+\lambda} < 1 \tag{32}$$

for all values of λ encountered so that the solution of (25)

$$V^{i} = V^{0}\left(\frac{1-\lambda}{1+\lambda}\right)^{i} \tag{33}$$

remains bounded as i increases. This means that the finite difference scheme considered here is always stable, regardless of the size of Δt.

R.P. Eddy

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