SOME TWO SAMPLE TESTS BASED ON ORDERED OBSERVATIONS FROM THE EXPONENTIAL DISTRIBUTION

BY

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1. Introduction

Let \( x_{11} \leq x_{12} \leq \cdots \leq x_{1n_1} \) and \( x_{21} \leq x_{22} \leq \cdots \leq x_{2n_2} \) be two random samples \( (S_{n_1} \text{ and } S_{n_2}) \) from populations having p.d.f.'s \( f(x; \lambda_1, \theta_1) \) and \( f(x; \lambda_2, \theta_2) \) respectively, where

\[
f(x; \lambda, \theta) = \frac{1}{\theta} \exp \left( \frac{-x}{\theta} \right).
\]

Let \( S_{r_1} \) and \( S_{r_2} \) be the sets of the first \( r_1 \) and \( r_2 \) smallest observations of \( S_{n_1} \) and \( S_{n_2} \) respectively. Then the p.d.f.'s of \( S_{r_1} \) and \( S_{r_2} \) are given, say, by

\[
g(x_{11}, \ldots, x_{1r_1}; \lambda_1, \theta_1) \quad \text{and} \quad g(x_{21}, \ldots, x_{2r_2}; \lambda_2, \theta_2),
\]

where

\[
g(x_1, x_2, \ldots, x_r; \lambda, \theta) = \frac{n!}{(n-r)!} \frac{1}{\theta^r} \exp \left\{ -\frac{r}{\theta} \left[ \frac{\sum_{i=1}^{r} (x_i - \lambda) + (n-r)(\lambda - \lambda)}{\theta} \right] \right\}
\]

The likelihood ratio tests based on the complete sets, \( S_{n_1} \) and \( S_{n_2} \), are special cases of those obtained by Sukhatme [2,3]. It can be shown that similar likelihood ratio tests based on \( S_{r_1} \) and \( S_{r_2} \) may be obtained by following Sukhatme's procedure [2]. In this report these likelihood ratio tests are reduced to equivalent tests which are expressed in terms of the well-known Chi-square and Snedecor's F distributions. Furthermore, some of the tests obtained in this report can be extended to \( k \)-sample tests.
Since percentage points for the \( \chi^2 \) and \( t \) distributions are tabulated, tests involving these random variables are useful in application. We remark that the likelihood ratio test for the hypothesis \( H_2 \) (see Section 3) has been obtained by Paulson [1].

2. Preliminary lemmas.

We give several lemmas which were used to obtain the distributions of the reduced statistics. Lemmas 1-3 can be proved by the use of characteristic functions and their proofs are omitted. Proofs of lemmas 4-9 are given.

In lemmas 1, 2 and 3 below, we let \( x_1 \leq x_2 \leq \ldots \leq x_r \leq \ldots \leq x_n \) be a random sample from a population having p.d.f. (1) and we define statistics \( u, v, \) and \( h \) as,

\[
(3) \quad u = \frac{2}{\theta} \left[ \sum_{i=1}^{r} (x_i - \lambda) + (n - r)(x_r - \lambda) \right].
\]

\[
(4) \quad v = \frac{2}{\theta} \left[ \sum_{i=1}^{r} (x_i - x_1) + (n - r)(x_r - x_1) \right].
\]

\[
(5) \quad h = \frac{2v}{\theta} (x_1 - \lambda).
\]

Lemma 1. \( u \) is distributed as \( \chi^2(2r) \).

Lemma 2. \( v \) is distributed as \( \chi^2(2r-2) \).

Lemma 3. \( v \) and \( h \) are independently distributed as \( \chi^2(2r-2) \) and \( \chi^2(2) \) respectively.

Lemmas 1-3 deal with the case of two samples. The statistics \( u_1, v_1 \) and \( u_2, v_2 \) are defined as in (3) and (4). Three additional variables \( w_1, w_2, \) and \( w \) are defined in (6); \( (\cdot)^2 := \langle \cdot \rangle^2 \).

\[
(6) \quad w_1 = \frac{2\theta}{\xi_1} (x_{11} - x_{12}), \quad \text{for} \quad x_{11} > x_{12}
\]
\( z_2 = \frac{n_2}{\theta_2} (x_{21} - x_{11}) \), for \( x_{21} > x_{11} \)

(8) \( w = w_2 \), when \( x_{11} > x_{21} \) and \( w = w_2 \), when \( x_{21} > x_{11} \).

**Lemma 4.** If \( A_1 = A_2 \), then

\[ \Pr(x_{11} > x_{21}) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2} \]

and

\[ \Pr(x_{21} > x_{11}) = \frac{n_1/\theta_1}{n_1/\theta_1 + n_2/\theta_2} \]

**Proof:**

\[ \Pr(x_{11} > x_{21}) = \int_{A_1} \int_{A_1} \frac{n_1 n_2}{\theta_1 \theta_2} e^{-\frac{n_1}{\theta_1} (x_{11} - A_1)} - \frac{n_2}{\theta_2} (x_{21} - A_1) \, dx_{11} \, dx_{21} \]

\[ = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2} \]

Hence,

\[ \Pr(x_{21} > x_{11}) = 1 - \Pr(x_{11} > x_{21}) = \frac{n_1/\theta_1}{n_1/\theta_1 + n_2/\theta_2} \]

**Lemma 5.** If \( A_1 = A_2 \), then both \( w_1 \) (given that \( x_{11} > x_{21} \)) and \( w_2 \) (given that \( x_{21} > x_{11} \)) are distributed as \( \chi^2(2) \).
Proof. Since \( A_1 = A_2 \), \( w_1 \) can be written as

\[
  w_1 = \frac{2n_1}{\theta_1} \left[ (x_{11} - A_1) - (x_{21} - A_2) \right].
\]

Consequently,

\[
  x_{11} - A_1 = \frac{\theta_1}{2n_1} w_1 + (x_{21} - A_2).
\]

Let \( x_{11} - A_1 = y_1 \) and \( x_{21} - A_2 = y_2 \), then the condition that \( x_{11} > x_{21} \) is equivalent to \( y_1 > y_2 \). Since the joint distribution of \( y_1 \) and \( y_2 \) is, say

\[
  f(y_1, y_2) = \frac{n_1 n_2}{\theta_1 \cdot \theta_2} e^{-\frac{n_1}{\theta_1} y_1 - \frac{n_2}{\theta_2} y_2}, \quad y_1, y_2 > 0,
\]

we have

\[
  \Pr(w_1 \leq w | y_1 > y_2) = \frac{\Pr(w_1 \leq w, y_1 > y_2)}{\Pr(y_1 > y_2)}. \tag{12}
\]

According to lemma 1:

\[
  \Pr(y_1 > y_2) = \Pr(x_{11} > x_{21}) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2}.
\]

Further, it is readily verified that

\[
  \Pr(w_1 \leq w, y_1 > y_2) = \frac{n_2/\theta_2}{n_1/\theta_1 + n_2/\theta_2} \left[ 1 - e^{-w/2} \right]. \tag{13}
\]

Therefore,

\[
  \Pr(w_1 \leq w | y_1 > y_2) = 1 - e^{-w/2}. \tag{14}
\]
But, by (11)

\[
\Pr(w \leq w_{10}, x_{11} > x_{21}) = \Pr(x_{11} > x_{21}) \left[ 1 - e^{-\frac{w}{2w_{10}}} \right]
\]

and by lemma 2, \(\Pr(v_1 \leq v_{10})\) and \(\Pr(v_2 \leq v_{20})\) are cumulative \(\chi^2\)-distributions with \((2r_1 - 2)\) and \((2r_2 - 2)\) d.f.'s. Thus lemma 7 is proved.

**Lemma 8.** If \(A_1 = A_2\) then \(V_1, V_2\) and \(w\) are independently distributed as \(\chi^2(2r_1 - 2), \chi^2(2r_2 - 2)\) and \(\chi^2(2)\) respectively.

**Proof.** Since

\[
\Pr(v_1 \leq v_{10}, v_2 \leq v_{20}, w \leq w_0) = \Pr(v_1 \leq v_{10}, v_2 \leq v_{20}, w \leq w_0, x_{11} > x_{21}) + \Pr(v_1 \leq v_{10}, v_2 \leq v_{20}, w \leq w_0, x_{11} < x_{21})
\]

then by (17) lemma 8 follows.

**3. Likelihood ratio tests and equivalent reduced tests.**

The various hypotheses and their associated likelihood ratio and equivalent reduced tests are listed below. The details involved in obtaining the likelihood ratio will not be given here, since they are well known.

**A. Statement of hypotheses:**

a) \(H_1:\) To test \(\theta_1 = \theta_2\)

(assuming \(A_1\) and \(A_2\) are known).

b) \(H_2:\) To test \(\theta_1 = \theta_2\)

(assuming \(A_1 = A_2\)).

c) \(H_3:\) To test \(\theta_1 = \theta_2\).

\(H_4:\) To test \(\theta_1 = \theta_2\)

(assuming \(\theta_1\) and \(\theta_2\) are known).
a) $H_0$: To test $A_1 = A_2$
(assuming $\theta_1 = \theta_2$).

f) $H_6$: To test $A_1 = A_2$.

g) $H_7$: To test $A_1 = A_2$ and $\theta_1 = \theta_2$.

B. Likelihood ratio tests

In a), b) and c) below we let

$$k = \frac{2}{\chi^2} \left( \frac{r_1 + r_2}{r_1} \right)^{r_1}$$  \hspace{1cm} (19)

a) For $H_1$:

$$\lambda_1 = k \left[ (1 + c_1) r_1 (1 + \frac{1}{c_1} r_2) \right]^{-1}$$  \hspace{1cm} (20)

where

$$c_1 = \frac{\sum_{j=1}^{r_2} (x_{2j} - A_2) + (n_2 - r_2)(x_{2r_2} - A_2)}{\sum_{j=1}^{r_1} (x_{1j} - A_1) + (n_1 - r_1)(x_{1r_1} - A_1)}$$  \hspace{1cm} (21)

b) For $H_2$:

$$\lambda_2 = k \left[ (1 + c_2) r_1 (1 + \frac{1}{c_2} r_2) \right]^{-1}, \text{ if } x_{11} < x_{21}$$

$$\lambda_2 = k \left[ (1 + \frac{1}{c_2}) r_1 (1 + c_2 r_2) \right]^{-1}, \text{ if } x_{21} < x_{11}$$

(22)

where
e) For $H_5$:

\begin{align}
\lambda_5 &= (1 + c_5)^{-(r_1 + r_2)} \quad \text{if } x_{11} > x_{21} \\
&= (1 + c_5)^{-(r_1 + r_2)} \quad \text{if } x_{11} < x_{21}
\end{align}

where

\begin{align}
c_5 &= \frac{\sum_{i=1}^{2} x_{ii} - x_{21}}{r_1 \sum_{i=1}^{2} x_{ii} - x_{21} + (n_i - r_i)(x_{i1} - x_{i11})} \\
&\quad + \sum_{i=1}^{2} \frac{r_i}{(n_i - r_i)(x_{i1} - x_{i11})}
\end{align}

f) For $H_6$:

\begin{align}
\lambda_6 &= (1 + c_6)^{-r_1} \quad \text{if } x_{11} > x_{21} \\
&= (1 + c_6)^{-r_2} \quad \text{if } x_{11} < x_{21}
\end{align}

where
\[ c_6 = \frac{n_1(x_{11} - x_{21})}{\sum_{j=1}^{r_1} (x_{1j} - x_{11}) + (n_1 - r_1)(x_{1r_1} - x_{11})} \]  

\[ c_6 = \frac{n_2(x_{21} - x_{11})}{\sum_{j=1}^{r_2} (x_{2j} - x_{21}) + (n_2 - r_2)(x_{2r_2} - x_{21})} \]  

(31)

g) For \( h_7 \):

\[ \lambda_7 = \frac{2}{\prod_{i=1}^{r_1} \left( \frac{\hat{\xi}_i}{\xi_i} \right)} \]

where

\[ \hat{\xi}_i = \frac{1}{r_1} \sum_{j=1}^{r_i} (x_{ij} - x_{i1}) + (n_i - r_i)(x_{i1} - x_{i1}) \]

\[ \hat{\theta} = \frac{1}{r_1 + r_2} \sum_{i=1}^{r_i} \left[ \frac{r_i}{\sum_{j=1}^{r_i} (x_{ij} - \hat{\lambda}) + (n_i - r_i)(x_{i1} - \hat{\lambda})} \right] \]

and where \( \hat{\lambda} = \min (x_{11}, x_{21}) \).

C. Reduced Tests

By the use of the lemmas in section 2, \( \lambda_1, \lambda_2, \ldots, \lambda_6 \) can be reduced to the following equivalent tests having the corresponding distributions (see Table 1). The authors have not succeeded in reducing \( \lambda_7 \) to a known test.
<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Equivalent reduced Tests</th>
<th>Distributions</th>
<th>Critical region</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>$f_1 = r_1 \frac{v}{r_2} c_1$</td>
<td>$F(2r_2, 2r_1)$</td>
<td>(2)</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$f_2 = \frac{r_1-1}{r_2} c_2$, if $x_{11} &lt; x_{21}$</td>
<td>$F(2r_2, 2r_{1-2})$</td>
<td>(2)</td>
</tr>
<tr>
<td></td>
<td>$f'<em>2 = \frac{r_2-1}{r_1} c'<em>2$, if $x</em>{21} &lt; x</em>{11}$</td>
<td>$F(2r_1, 2r_{2-2})$</td>
<td>(2)</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$f_3 = \frac{r_1-1}{r_2-1} c_3$</td>
<td>$F(2r_2-2, 2r_{1-2})$</td>
<td>(2)</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$f_4 = c_4$</td>
<td>$\chi^2 (2)$</td>
<td>(1)</td>
</tr>
<tr>
<td>$H_5$</td>
<td>$f_5 = \frac{2r_1 + 2r_2 - 4}{2} c_5$, if $x_{11} &gt; x_{21}$</td>
<td>$F(2, 2r_1 + 2r_2 - 4)$</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>$f'<em>5 = \frac{2r_1 + 2r_2 - 4}{2} c'<em>5$, if $x</em>{21} &gt; x</em>{11}$</td>
<td>$F(2, 2r_1 + 2r_2 - 4)$</td>
<td>(1)</td>
</tr>
<tr>
<td>$H_6$</td>
<td>$f_6 = \frac{2r_2 - 2}{2} c_6$, if $x_{11} &gt; x_{21}$</td>
<td>$F(2, 2r_1-2)$</td>
<td>(1)</td>
</tr>
<tr>
<td></td>
<td>$f'<em>6 = \frac{2r_2 - 2}{2} c'<em>6$, if $x</em>{21} &gt; x</em>{11}$</td>
<td>$F(2, 2r_1-2)$</td>
<td>(1)</td>
</tr>
</tbody>
</table>
In Table 1 numbers in the "critical regions" column indicate that the reduced tests obtained may be either one-sided or two-sided. For example, consider the case where \( r_1 = r_2 = 10 \) and \( \alpha = .05 \). Then for the various \( H_i \), \( i = 1, 2, 3, 4, 5, 6 \) we have the following critical regions which are summarized for convenience in Table 2.

### TABLE 2

**Critical Regions**

<table>
<thead>
<tr>
<th>( H_i )</th>
<th>( f_1 &gt; 2.44 ) or ( f_1 &lt; \frac{1}{2.44} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_2 )</td>
<td>( f_2 &gt; 2.56 ) or ( f_2 &lt; \frac{1}{2.56} ) when ( x_{12} &lt; x_{21} )</td>
</tr>
<tr>
<td></td>
<td>or ( f_2' &gt; 2.56 ) or ( f_2' &lt; \frac{1}{2.56} ) when ( x_{22} &lt; x_{11} )</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>( f_3 &gt; 2.60 ) or ( f_3 &lt; \frac{1}{2.60} )</td>
</tr>
<tr>
<td>( H_4 )</td>
<td>( f_4 &gt; 5.99 )</td>
</tr>
<tr>
<td>( H_5 )</td>
<td>( f_5 &gt; 3.26 ) when ( x_{11} &gt; x_{21} )</td>
</tr>
<tr>
<td></td>
<td>( f_5' &gt; 3.26 ) when ( x_{21} &gt; x_{11} )</td>
</tr>
<tr>
<td>( H_6 )</td>
<td>( f_6 &gt; 3.57 ) when ( x_{11} &gt; x_{21} )</td>
</tr>
<tr>
<td></td>
<td>( f_6' &gt; 3.57 ) when ( x_{21} &gt; x_{11} )</td>
</tr>
</tbody>
</table>
References:


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