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Analysis of the Hydrogen Eclipse Observations to Determine the Thermodynamic State of the Solar Chromosphere

by
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The following report was prepared by the authors as a part of their work in preparation for the eclipse expedition of High Altitude Observatory to the Sudan on 25 February 1952. It was written before the eclipse. We issue it now as we prepare to use the methods outlined for reduction of the spectra obtained. We express our thanks to the authors and to the University of Utah for the many forms of assistance they have provided to us in this work. Without this aid we would have found it impossible to make sound plans for the expedition in the short time that was available to us.

* * * * * *

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Eugene N. Parker and Richard N. Thomas

I. Introduction

The following discussion contains an outline of, and a justification for, the theoretical analysis of the data to be obtained at the 1952 eclipse. Actually, the following report considers only a limited portion of the expected data -- that relating to the hydrogen Balmer and Paschen series, and the Balmer continuum. An analysis of earlier eclipse observations demonstrated the feasibility of such an analysis. On the basis of the results from this analysis, we suggested the advisability of an eclipse expedition, with limited but specific objectives, aimed only at the data required for this analysis. We proposed deliberately foregoing the usual attempts to measure all possible phenomena associated with the eclipse. The 1952 eclipse program, undertaken by High Altitude Observatory in close cooperation with and supported by the Naval Research Laboratory, had somewhat wider objectives, but the effort was concentrated on the
objectives outlined here. Dr. John P. Hagen and his associates planned to conduct simultaneous radio observations at two frequencies during the eclipse. It was hoped thereby to obtain some resolution of the current dilemma between radio and optical observations of the outer solar atmosphere. The optical observations were to be made by High Altitude Observatory, and the analysis to be carried out in close cooperation with the Utah group. The radio observations were to be handled by the Naval Research Laboratory group. The following discussion relates only to the analysis of the hydrogen observations originally proposed.

The general philosophy of the program to be discussed is one of an attempt to investigate certain gross features of the hydrogen spectrum that may be unambiguously interpreted to specify certain gross features of the structure of the chromosphere. Since the solar chromosphere presents so many observational anomalies, such attempts may be overly optimistic. Nonetheless, by restricting ourselves to hydrogen we may hope, because of its preponderant abundance, to establish results characterizing the thermodynamic structure of the chromosphere. Furthermore, by considering only relative spectral intensities, we may hope to avoid the usual difficulties encountered with absolute standards. If the procedure is successful in establishing these gross features, we may then proceed to analyze the full details of the spectrum to obtain the full details of the state of the chromosphere. There is a certain amount of risk in attempting to so divide the problem of the state of the chromosphere -- one recalls attempts to predict the temperature of the planetary nebulae from the action of the radiation field of the exciting star on the hydrogen alone without considering the thermostat action of the oxygen "impurities." It is thus essential to examine critically all data, not only that for hydrogen, seeking contradictions to the present procedure. The difficulty in such examinations -- and several have been offered as, for example, the criticism of the high temperature model because of the lack of metallic metastable lines -- lies in an inability to evaluate the exact situation to be expected until the gross thermodynamic structure is known. We attempt then to obtain this gross structure, selecting methods general enough to remove the above possibilities of internal inconsistency.

II. Method of Reduction.

A. The Balmer and Paschen decrements.

Specifically, one observes during an eclipse all the light originating above a certain height in the chromosphere. Let us consider, then, the energy emitted by transitions from the nth to the second level of the hydrogen atom in terms of the state of the chromosphere. We consider one cm$^3$ of matter at $(x, y)$ in the chromosphere.
Figure I

$w$ is the distance from $(x, y)$ to the surface of the sun. We shall be specifically interested in ascertaining what characteristics of the thermodynamic state can be varied to give agreement between the theoretical and the observed Balmer decrement. One finds that on the basis of thermodynamic equilibrium and an atmosphere transparent to its own radiation, the computed decrement is in every case too high. Thus let us compute the energy per unit solid angle $E_n h$ with all possible variations of the above, evidently over simplified, assumptions included. We include, therefore, departures from thermodynamic equilibrium by introducing the standard $b_n$ defined in equation (6) below, include self-absorption effects, and make no attempt to specify the electron temperature, $T_e$. Thus, in standard notation,

$$d\epsilon_n h = b_n(x, y) h v_n^2 A_n^2 \frac{1}{4\pi} dv$$  \hspace{1cm} (1)$$

where $N_n$ is the number of particles/cm$^3$ at $x, y$ and we are considering the radiation from the element of volume $dv$. It will be easiest if we consider the radiation from a slab of width 1 cm in
the a direction. Thus, we operate with \( \int dz \) on (1) and we obtain

\[
d_{0}^{2}e_{n2} = \frac{N_{n}(x,y)h\gamma_{n}n_{2}A_{n}n_{2}}{4\pi^{2}} \int dx dy
\]  

(2)

The phenomenon of self absorption, of course, decreases this energy so that only the amount

\[
d_{x}^{2}e_{n2} = d_{0}^{2}e_{n2} \cdot \exp \left\{ -a_{2} \int_{y}^{\infty} N_{2}(x,y) dx \right\}
\]  

(3)

finally emerges from the atmosphere. We write

\[
N(x,y) = \int_{y}^{\infty} N_{2}(n,y) dx
\]  

(4)

so that

\[
d_{x}^{2}e_{n2} = d_{0}^{2}e_{n2} \cdot e^{-a_{2}N(x,y)}
\]  

(5)

Now, using \( X_{n} = \frac{N_{n}}{N_{\infty}} \) where \( N_{n} \) is the energy to ionize from the nth level (assuming hydrogen sufficiently ionized that \( N_{1} = N_{\infty} \)):

\[
\text{Thus the total energy emitted is}
\]

\[
E_{n2} = \int_{0}^{\infty} d_{x}^{2}e_{n2} = \frac{h\gamma_{n}n_{2}}{4\pi^{2}} \left( \frac{h^{2}}{2\pi^{2}k} \right)^{3/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} b_{n}^{2}e_{n_{2}} \cdot e^{-3/2\cdot X_{n} \cdot e^{-a_{2}N(x,y)}} dx dy
\]  

(7)

where \( C_{1} \) is a numerical constant. The integral may be thought of as a weighted average of the emission from hence the population of the nth level. We find it convenient therefore to also use in the discussion equation (7) without the substitution (6):

\[
E_{n2} = C_{1} \gamma_{n}n_{2} \int_{0}^{\infty} b_{n}^{2}e_{n_{2}} \cdot e^{-3/2\cdot X_{n} \cdot e^{-a_{2}N(x,y)}} dx dy
\]  

(7)
Consider now equation (7). The Balmer decrement corresponds to a differentiation, or differencing, of equation (7) with respect to the quantum number $n$. Performing this operation we obtain:

\[
\frac{A_n}{b_n} = \left(\ldots\right) \sum_n \left[ \frac{A_n b_n}{n} + \frac{2 \Delta_n}{n} \right] \cdot A_n \left(\ldots\right) \sum_n \left[ \ldots\right]
\]

where the inequality sign refers to the whole term, including the accompanying sign. We have also taken the absorption coefficient $a_{n2}$ to vary as $n^{-3}$. The dotted brackets in each case contain a positive quantity. The result on the $b_n$ comes only from the assumption that $T_e > T_{\text{eff}}$ for the sun — and independent of the numerical value of $T_e$. We see thus that only the third term gives a negative contribution to the right hand side. Thus only the self-absorption effect can remedy the difficulty mentioned above — that the observed Balmer decrement is smaller than the thermodynamic equilibrium value. The effect of departure from thermodynamic equilibrium only increases the discrepancy, unless one wishes to adopt $T_e < T_{\text{eff}}$, a hardly plausible result these days. Furthermore, we shall in the following section (III A 1) demonstrate from the observations that $b_n > b_{n+1}$.

We recognize now that the self-absorption is a function of one parameter only, the population of the two-quantum level. Hence we should be able to use the observed Balmer decrement in conjunction with the result from equation (9) to obtain this population of the two-quantum level as a function of height. Since we do not a priori know the thermodynamic structure of the atmosphere, the procedure must necessarily be one of successive approximation. In the analysis earlier mentioned, only the first approximation was carried out, i.e., an isothermal chromosphere was assumed and departures from thermodynamic equilibrium neglected. Actually, we note from equation (8) that most of the emission comes from the lower, central part of the atmospheric region observed; so the first approximation is fairly accurate. The plans for the forthcoming eclipse indicate that considerably more observational data will result, so the successive approximation scheme will be necessary to analyze the data.

Consider the first approximation. For constant $T_e$ and $b_n = 1$, we have an exponential gradient for $N_e$ which we may write:
\[ N_2^2 = N_0^2 e^{-\beta x} - \beta y^2 \]  \hspace{1cm} (10)

according to the usual geometric approximation for \( y \leq r \). The contribution of large \( y \) to the chromospheric observations is negligible. We have now two alternative procedures to carry out the computation.

In the first, we substitute (10) in both equation (4) and equation (7), and carry out the integration in series form. We obtain:

\[ E_{n2} = (2C_T)^{-\frac{3}{2}} N_0^2 \frac{\hbar}{\sqrt{n}} \left[ 1 + \sum_{n} \frac{(-1)^m (N_{n2}^{m-1})^2}{2^{2m-1}(n-2)} I_2 + \ldots \right] \]  \hspace{1cm} (11)

where \( N_{n2} \) is the number of second-quantum level atoms along the line of sight and:

\[ I_j = \frac{2}{r^2} \int_{0}^{\infty} e^{-s^2} g_j(s^2) ds \quad g(s^2) = \frac{2}{r^2} \int_{0}^{\infty} e^{-s^2} ds \]  \hspace{1cm} (12)

\( C_2 \) is a constant.

The expression (11) represents the thermodynamic equilibrium expression save for the last bracket, which represents the self-absorption. The expression is convenient to use for small absorption (small \( N_{2n2} \)) -- it does not converge for large \( N_{2n2} \) values. For the higher Balmer lines, however, satisfactory values for \( N_{2n2} \) may be quickly obtained by:

\[ \log \left[ \frac{1 + \sum \ldots (N_{2n2}^{m-1})}{1 + \sum \ldots (N_2^{m-1})} \right] \text{ observed decrement } - \Delta_n \log \left[ \frac{\hbar \nu n_2^2 \tau_2}{\hbar \nu n_2^2 \tau_2} \right] \]  \hspace{1cm} (13)

We have necessarily omitted the term \( \chi \), since it depends upon the (unknown) value of \( T_e \). We note however that the omission is trivial for high enough \( n \) -- and we prefer the later Balmer lines anyway to avoid convergence difficulty with the series expression for the absorption. The left side of equation (13) may be reduced to a function of a single variable by reducing \( a(n-k)^2 \) to \( a_n^2 \) using an \( n^{-3} \) dependence for \( a_n^2 \). This procedure was followed in reference (1).
Alternatively we may proceed from equation (8). Recognizing that we may write the emission coefficient as:

$$j_\nu = A_n \nu^2 \frac{N_n}{h}$$  \hfill (14)

and the absorption coefficient:

$$k_\nu = a_n N_n^2$$  \hfill (15)

with the associated tangential optical depth:

$$d\tau_\nu = a_n N_n^2 dx$$  \hfill (16)

we may write (8):

$$E_n = \frac{1}{4\pi} \int_0^T \int \frac{j_\nu}{k_\nu} e^{-\tau_\nu} d\tau_\nu dy$$  \hfill (17)

We note now the expression (15) must include induced emissions to be complete - so that we may write:

$$a_n = a_n^1 N_n \left(1 - \frac{N_n}{N_2} \frac{\omega_n^2}{\nu_n^2} \right)$$  \hfill (18)

and use the relation:

$$\frac{A_n}{a_n^1 N_2} = \frac{8\pi \nu^2}{c^2} \frac{\omega_n^2}{\nu_n^2}$$  \hfill (19)

whence:

$$\frac{j_\nu}{k_\nu} = \frac{2\nu^3}{c^2} \frac{\omega_n^2}{\nu_n^2} \frac{N_n}{N_2} \left[1 - \frac{N_n}{N_2} \frac{\omega_n^2}{\nu_n^2}\right]^{-1}$$

$$= \frac{2\nu^3}{c^5} \left[\frac{b_n^2}{b_n^2} e^{h \nu / kT_n} - 1\right]^{-1}$$  \hfill (20)

using equation (6).
Alternatively then we may write (17) as:

\[ \frac{2n^2}{c^2} - \frac{2h \nu^3}{e} \int_0^{\infty} \left[ \frac{b^2}{b_n} e^{h \nu/kT} - 1 \right]^{-1} e^{-\frac{\nu}{kT}} d\nu \, dx \]  

(17.1)

We write (17) in both forms to indicate the nature of the approximation made. In the usual discussion, \(j_\nu/kT\) is taken as function of the temperature only - in which case we could remove it from under the integral in (17). Actually, however, the ratio depends upon \(N_0\) as well as \(T\). In discussing the first method of analysis we have taken \(b_n = 1\). There, however, we were concerned only with the higher \(n\) values. A variation in \(b_2\) entered as a second order correction through the integral of the exponential. In (17.1), however, \(b_2\) enters directly. The approximation that \(b_n \sim 1\) for large \(n\) is much better than that for \(b_2 \sim 1\). Thus, combining these two results - the first order appearance of \(b_2\) and the likelihood of its considerable departure from unity, it does not seem too satisfactory an approximation to remove \(j_\nu/kT\) from the integral. We note, however, that we shall be differentiating the result - and so some of the error will disappear. Thus we proceed on this approximation and obtain:

\[ \frac{2n^2}{c^2} - \frac{2h \nu^3}{e} \int_0^{\infty} \left[ \frac{b^2}{b_n} e^{h \nu/kT} - 1 \right]^{-1} \left( 1 - e^{-\gamma(x)} \right) dx \]  

(21)

We note that this expression complements that obtained by the first method - for as \(T\) becomes sufficiently great the emission per unit area - i.e., ignoring the last integration over the atmosphere lying above a given point - approaches the Planck function modified by the \(b_n/b_2\) factor. Thus we have an expression suitable for the case of high absorption, when the series expansion in equation (11) fails to converge.

The actual evaluation of the integral in (21) must be performed numerically - for we note that \(\gamma(x)\) is the \(N_2d_2h\) of the previous method and decreases exponentially with height. For small \(\gamma\), the exponential may be expanded and the integral evaluated as:

\[ \int_0^{\pi} \left[ 1 - e^{-N_2(h)\gamma} \right] dx = \left( N_2(h) \right) \gamma \left[ \frac{N_2(h) \gamma}{2} \right] + \left( N_2(h) \gamma \right)^2 \left[ \frac{N_2(h) \gamma}{3} \right] \left( 1 - e^{-N_2(h)\gamma} \right) \]

so that this expansion may be used if preferred.

The last procedure discussed is actually of more value in the second and higher approximations, where some idea of the variation of \(j_\nu/kT\) with \(\gamma\) is had. It is largely for this reason that we have discussed it. If desired however, equations (21) and (22) may be used in place of equation (11).
From the above procedures, then, we obtain first approximations values of \( N_2(h) \) from the Balmer decrement. Clearly the discussion may be taken over intact to observations of the Paschen decrement, obtaining a check on the \( N_2 \) values. For the higher order approximations, we require a knowledge of the \( N_e \) and \( T_e \) values in order that the isothermal and \( b_n = 1 \) assumptions may be dropped. Clearly, some information on these quantities should result from the knowledge just gained of \( N_2(h) \). There are, in addition, several other sources of information. We turn to consider them.

B. Determination of \( N_e \) and \( T_e \)

We have available three distinct methods of estimating various ones of the unknown quantities \( N_e, V_A, N_2, T_e, V_B \) from the eclipse optical observations. One method represents an application and extension of the results of Section A. Because it is an extension, and because it was developed simultaneously with that method, we consider it first. The method gives information on \( N_e, V_A, N_2, T_e \) and \( V_B \). The second method rests on a study of line profiles and gives information on \( T_e \) and \( N_e \) at each height for which precise observations exist. The third, and possibly least accurate method, leads to values of \( T_e \).

1. The apparent emission height-gradient.

Consider two methods commonly used in obtaining the height gradient of hydrogen in the solar atmosphere. In the first method we compare the emission from a given Balmer line at two eclipse heights. With the exception of an error resulting from the neglect of self-absorption effects, the emission height-gradient may be interpreted directly as the height gradient of the number of atoms in the upper level of the line. The neglected self-absorption effects clearly make this inferred value a lower limit to the true value. In the second method we attempt to eliminate self-absorption effects by observing the height at which the various Balmer lines just vanish. Presumably, the various lines reach the same intensity at their respective disappearing points, and the emission is so small that self-absorption effects drop out. By correcting for the difference in transition probability, the comparison of vanishing points for the two lines gives the ratio of the population of the two levels at the two heights. The problem is to convert this population ratio into a height gradient for the population of either level. From equation (6) we see that, if the \( b_n \) values were the same for the two levels at a given height, the figure just obtained would, when corrected for the Boltzmann factor \( \exp \left( \frac{A_e}{h} \right) \), give the height gradient of either level. If
however, $b_n > b_{n+k}$, then the emission at a given height for Balmer line $H_n$ would be greater than it would be in the case of $b_n = b_{n+k}$; the line $H_n$ would persist to a greater height relative to $H_{n+k}$, and we would underestimate the height gradient if we used the observational material with no $b_n$ correction. If $b_n < b_{n+k}$, the reverse situation would hold. We can then compare the density gradients obtained by the first and second methods, in the hope of obtaining the behavior of $b_n$. We note, of course, that a result showing the density gradient from the first method to be less than that from the second would be somewhat inconclusive because of the neglected self-absorption effect. Fortunately, however, we find the first-method gradient to exceed the second. The situation is not peculiar to data from one eclipse, nor is the difference so small as to be masked by observational error. (The difference is almost a factor of 2.) Hence it would appear that, from this comparison of two methods of estimating the hydrogen density gradient, we have a conclusive demonstration of the direction of deviation of the chromosphere from thermodynamic equilibrium; that is $b_n > b_{n+k}$.

Thus, we have demonstrated $b_n > b_{n+k}$, the results needed for the Section A analysis of self-absorption. We proceed now to make further use of the apparent density gradient.

We have remarked that the apparent emission gradient differs from the true density gradient by the self-absorption effect. We can, however, correct for this self-absorption by using the results from Section A. Then the corrected results give the true height gradient of $N_n$, and we can write for each $N_n, n > 2$, at each height, from equation (6):

$$
\frac{d \ln N_n}{dh} - \frac{d \ln N_n^2}{dh} - \frac{d \ln \frac{T_e}{T_a}}{dh} \left[ \frac{3}{2} + \frac{3}{n} \right] + \frac{d \ln b_n}{dh}
$$

where the left side is known from the above corrected emission gradient. We can also write this equation for $N_2$, using the results from Section A to obtain the left side. Thus we have a set of equations, as many as we have observed hydrogen lines, Balmer or Paschen, with the unknowns $\nabla^2 \ln N_n, \nabla^2 \ln T_e, \nabla^2 \ln b_n$, and the $\nabla^2 \ln b_n$, while the $\nabla^2 \ln b_n$ are obviously not all independent, the dependence cannot be computed until $N_2, T_e$ are known. Thus a preliminary solution of the equation (23) must be carried through ignoring the $b_n$ term. The error is greatest in the $b_2$ term, the $N_2$ equation; so it is preferable to carry through a solution neglecting the $N_2$ equation if possible. Then the second approximation to both this procedure and that of Section A may be made.
we note, however, that neither this Section B 1, nor
Section A provide values of $N_e$ directly. One can estimate
a value by successive approximations on a consistency basis,
but it is useful to have more direct methods in the first
approximation.

Before proceeding to the second method we note that
the preceding discussion enables a test of the validity of
the hydrostatic equilibrium to be made. We can write this
equation as:

$$C = \frac{d}{dh} \ln \left( \frac{N_e^2}{(1 - \eta)^2} \right) + 2 \frac{d \ln N_e}{dh} + 2 \frac{\mu_i^2}{R T_e} g \quad (24)$$

where $\eta$ is the fraction of the hydrogen yet unionized. Once
estimates of $T_e$ and $N_e$ are available, $\eta$ can be estimated.
The remaining quantities in (24) are known, and the validity
of the equation can be checked.

2. Results from line profiles.

There are two varieties of measures involving line
profiles. The first is a direct measure of a single line
profile. This technique has been used by R. V. Redman, 3)
who finds the early hydrogen lines and the the metallic
lines exhibit Doppler profiles. Redman plans to repeat
these measures at the 1952 eclipse. No such work is
contemplated in our plans. We shall, therefore, not
consider this method further. It is not meant thereby to
underemphasize the method, for on the contrary it is the
most reliable method for clearly indicating the kinetic
state of the atmosphere. Thus the method is critical in
any discussion of atmospheric kinetic temperature.

The second method dealing with line profiles usually
concerns the wings of the lines, where the effect of
collisional broadening becomes of importance. We note,
however, that for sufficiently high temperature the thermal
broadening can influence appreciably the merging of the
lines. We consider the problem.

The Stark broadening due to the positive ions causes
a merging of the lines at finite $n$. An approximate
relation, agreeing well with experiment, between the $n$
value for the last resolved line and the ion density has
been computed by Inglis and Teller: 4)

$$\log N_i = 23.26 - 7.5 \log n \quad (25)$$
The relation considers, however, only Stark broadening, and we wish also to include the Doppler broadening, given by the expression:

\[ a - \frac{(\psi - \psi_0)c}{\nu} \]  

(26)

Now, the line profile for the Stark effect will be of the form

\[ a^2 + \frac{a^2}{(\psi - \psi_0)^2} \]  

(27)

Thus, the Doppler and Stark effect together will yield

\[ a^2 \int_0^{\infty} \frac{e^{-\frac{2a^2}{a^2 + (\psi - \psi_c)^2}}}{a^2 + (\psi - \psi_c)^2} d\psi \]  

(28)

Unfortunately the result does not come out explicitly in any neat form. Therefore, noting that the results to be obtained are only approximate anyway because of the uncertainty as to when a line is exactly resolved or unresolved, we use a somewhat approximate method and thereby save much labor. We wonder what gaussian curve will approximate to (27) so far as resolution criteria are concerned. Clearly \( a \) is the width at half maximum in (27). We should like to replace (27) by

\[ e - \frac{(\psi - \psi_0)^2}{a} \]  

(29)

and find \( a \) in terms of \( a \). We shall adjust \( a \) so as to give the same resolution as (27).

We shall use the resolution criterion that for a gaussian distribution, the half-width at half-maximum is just one half \( \Delta \psi \) the line separation. Thus, for resolution,

\[ c = \frac{\Delta \psi}{2 \ln 2} \]  

(30)

We see then that midway between the lines the intensity is

\[ I_m = 2 \cdot \frac{1}{2} = 1 \]

and at the center

\[ I_c = 1 + e^{-\frac{\Delta \psi}{2 \ln 2}} \]  

(31)

Thus

\[ \frac{I_c - I_m}{I_m} = 6 \ln 1 - e^{-\frac{\Delta \psi}{2 \ln 2}} \]  

(32)
We now ask ourselves how $a$ is related to $\alpha$ so that $\alpha$ is the same when using (27) as when (28) is substituted for (27).

\[
\frac{I_0 - I_m}{I_m} = \left(1 + \frac{a^2}{\alpha^2 (2\Delta V)^2}\right) - 2 \frac{\alpha^2}{\alpha^2 + (\Delta V)^2} \left(\frac{1}{\alpha^2} + \frac{2}{(\Delta V)^2}\right)
\]

One finds, upon setting this equal to $6 \ln I$ (which is 1/16) that

\[
a^2 = \Delta V^2 \left(\frac{1}{2} + \alpha^2 (6 \ln I)\right)
\]

or

\[
a^2 = \frac{\Delta V^2}{\gamma_2} = a \gamma_2 \ln 2
\]

Thus, we see that as far as resolution is concerned, a dispersion profile with a half width at half maximum of $\alpha$, may be replaced by a gaussian with mean width $\alpha/\gamma_2 \ln 2$ or a half width at half maximum of $\alpha/\gamma_2$. Therefore, we write the combined Doppler and Stark profile as

\[
\phi = \frac{(\nu - \nu_0)^2}{(\nu_0 - \frac{\alpha^2}{2}) + \frac{a^2}{\gamma_2 \ln 2}}
\]

where $a$ is the half width at half maximum of the Stark profile.

Now $\alpha$ is independent of the ground state since it is only the higher states which are significantly perturbed by the Stark effect. Thus, for a given upper level $n$, the energy perturbation is $\Delta E_n$. And the resulting frequency spread is

\[
\Delta \nu_n = \frac{1}{\hbar} \Delta E_n
\]

We see then that the density broadening will be the same for the Balmer and Paschen series. We consider now the Doppler broadening.

\[
\sqrt{\nu_{nm}} = \frac{R}{\hbar} \left(\frac{1}{m^2} - \frac{1}{n^2}\right)
\]
Adjacent lines are separated by

\[ V_{n+1} = \frac{R}{n} \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = \frac{R}{n} \cdot \frac{2n+1}{n^2(n+1)^2} \]  

(36)

If we take a line as unresolved when it just falls to \( e^{-k^2} \) its maximum value for the frequency of the center of the next higher line, then for the two series with the ground states \( m \) and \( m' \) we have:

\[ k^2 = \frac{V_{n,n+1}^2}{(\frac{V_{n,n+1}}{c})^2 + (\frac{\Delta V_{n,n+1}}{2n})^2} \]  

(37)

\[ k^2 = \frac{V_{n',n'+1}^2}{(\frac{V_{n',n'+1}}{c})^2 + (\frac{\Delta V_{n',n'+1}}{2n'})^2} \]  

(38)

where \( n' - 1 \) and \( n - 1 \) are the last resolved lines of the \( m \) and \( m' \) series respectively. The subscripts in each case indicate the quantum numbers involved in the quantities. We use the results of Inglis and Teller to compute \( \Delta V_{n,n'} \). It should be noted that we cannot simply use their final results because, while we have merging of the lines as did they, part of our merging is a Doppler broadening. Thus, we wish to compute \( \Delta V_{n} \) for an unmerged line. They have that for a field \( F \), the energy spread in a level is

\[
\frac{1}{2} \text{ as } F
\]

\( a \) is the radius of the orbit and is \( n^2 a_0 \) where \( a_0 \) is the radius of the first Bohr orbit. Thus, the frequency spread is

\[ \Delta \nu = \frac{3aeF}{2h} \]

In terms of \( N \) the number of singly charged ions present Holtzmark 5) gives

\[ F = 3.7 \times N^{2/3} \]

Thus

\[ \Delta \nu = \frac{5.6 \times n^2 e^2 a_0 N^{2/3}}{2h} \]  

(39)
where $\mathcal{K} = \frac{5.6 \cdot 10^2}{h}$

Thus, using (40) we rewrite (37) and (38) as

$$
\psi_{n m}^2 \left( \frac{\psi}{c} \right)^2 + \frac{\hbar^2}{2 \ln 2} \frac{2 n \mathcal{K} \hbar^{2/3}}{n^2} = \frac{\psi_{n m}^2}{k^2} \tag{42}
$$

we wish to solve for $\psi$ and $N$. We find

$$
\frac{\psi}{c} = \frac{1}{k} \left( \frac{(n^2)^{1/4} - (n^2)^{1/4} - (n^2)^{1/4} - (n^2)^{1/4}}{(n^2)^{1/4} (1 - \frac{n^2}{n^2})^2} \right) \tag{43}
$$

$$
N = \left( \frac{\hbar^2}{k^2} \mathcal{K} \hbar^{2/3} \left( \frac{n^2}{n^2} \right)^{1/4} \left( 1 - \frac{m^2}{n^2} \right)^2 \right)^{3/4} \tag{44}
$$

or

$$
\frac{\psi}{c} \sim \frac{1}{k} \left( \frac{\hbar^4}{n^6} \left[ 1 - (\frac{n}{n})^{10} \right] \right)^{1/2} \tag{43.1}
$$

$$
N \sim \left( \frac{2 \hbar^2 \ln 2}{k^2 \mathcal{K} \hbar^2} \frac{1}{n^{10}} \left[ \frac{1 - (\frac{n}{n})^{10} (\frac{m}{m})^{10}}{1 - (\frac{n}{n})^{10} (\frac{m}{m})^{10}} \right] \right)^{1/4} \tag{44.1}
$$
If we use the resulting criterion cited above that the full width at half maximum is equal to $\eta_{n+1}$, then

$$e^{-k^2/4} = 1/2$$

$$k = 2\sqrt{2\ln 2} = 1.664$$

Further $k = 1.033$ cm$^2$/sec. Thus, given the last resolvable lines of Balmer and Paschen series, values for $N_e$ and $T_e$ result. (The value of $T_e$ follows directly from the value of $\eta/c$.) We note, incidentally, that the expressions (43.1) and (44.1) each fall into two parts, the second part in each being a correction factor of the order of unity involving each of the $n$, $n^2$, $m$, $m^2$ values. The other factor (43.1) involves only the Balmer series data $n$ and $m$ and is the expression that would be obtained were only thermal broadening included in the discussion. Thus, if the second factor were sufficiently near unity, the $T_e$ value would be fixed by the Balmer series alone. Similarly, the first factor in (44.1) depends only upon the Paschen series. We note, however, that the first factor is not the same as would be obtained by considering Stark broadening alone, but differs by $2^{-3/4}$. This factor has been introduced by the process of representing the Stark broadening in gaussian form. Had the half width at half-maxima for the gaussian representation been chosen the same as the half-width in the conventional expression (27), i.e., $a$ and not $a/\sqrt{2}$, the extra $2^{-3/4}$ factor would not appear in (44.1).

3. Measures in the continuum.

In the past, attempts have been made to compute the electron temperature by using the ratio of intensities at two frequencies in the Balmer continuum. However, considerable doubt is cast on these calculations by their failure to take into account the scattering by the electrons of the photospheric radiation. If $N_e$ is the electron density then the intensity at a given frequency due to both bound-free emission and scattering is

$$B_{\nu} = \#_1 N_e^2 e^{-h\nu/kT_e} + \#_2 N_e e^{-h\nu/kT_e} \nu^3$$

neglecting the small self-absorption effects.\(^1\) $\nu^3 e^{-h\nu/kT_e}$ represents the black body radiation from the photosphere of temperature $T$. If we take the ratio of $B_2$ for the two frequencies $\nu$ and $\nu'$, then

$$B_{\nu} = \#_3 N_e e^{-h\nu/kT_e} + \nu^3 e^{-h\nu/kT_e}$$

$$B_{\nu'} = \#_3 N_e e^{-h\nu'/kT_e} + \nu^3 e^{-h\nu'/kT_e}$$

\(^1\)
we see that
\[ \frac{E}{E'} = e^{\frac{h}{kT}} (\nu' - \nu) \quad \text{for large } N_e \quad (48) \]

and
\[ \frac{E}{E'} = e^{\frac{h}{kT}} \nu^3 (\nu' - \nu) \quad \text{for small } N_e \quad (49) \]

Clearly for pure scattering, i.e., small \( N_e \), the ratio is independent of height in the atmosphere except for very small geometrical dilution effects. As one goes higher in the atmosphere, we should then obtain more nearly (49). Thus, if one insists upon using (48) he writes

\[ T_e = \frac{\ln \frac{E}{E'}}{kL} \frac{\nu}{\nu'} \quad (50) \]

when actually he should write, if it is fairly high in the atmosphere:

\[ T_e = \frac{\ln \frac{E}{E'}}{k(\ln \frac{E}{E'} + 3 \ln \frac{\nu}{\nu'})} \quad (51) \]

Thus we see that the \( T_e \) values obtained from measures in the continuum, under the assumption that the observed radiation is chromospheric emission and not scattered radiation, are likely to be erroneous. A priori, with no knowledge of \( T_e \) and \( N_e \), it is impossible to predict the relative importance of the emission and scattering terms. Thus it seems the decision as to the utility of the method can be made best by comparing "apparent" values of \( T_e \) derived from equation (50). Since \( T_e \) presumably increases upward, and from (47) we see that the scattering increases in relative importance upward, the observed \( T_e \) will apparently decrease upward if the scattering is significant. Thus the utility of this third method is questionable, but a check on its utility is possible.

III. Conclusion

The foregoing provides an outline of the methods proposed to reduce the 1952 eclipse data. The results should provide values of \( N_e, T_e, \nabla T_e, \nabla N_e \) at a number of heights in the atmosphere. Hence, there are internal checks on the atmospheric structure. Thus, in view of the objective of trying to reconcile radio and optical data, one should from the foregoing have a consistent, well-defined set of results directly relevant to the fundamental problem.
IV. References.

5. Holtzmark, Phys. Z., 25, 73, 1924.

V. Acknowledgements.

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Approved for Submission as Technical Report

Walter Orr Roberts

17 July 1952
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